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Application of the Method of Trigonometrical Substitution in Solving Problems

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Abstract:
The use of trigonometric substitution in solving algebraic problems aims to establish a relationship between different branches of mathematics, namely: algebra and trigonometry. It is important to instill in students' courage and resourcefulness in finding ways to solve problems not only in the immediate environment of the conditions, but also in a wider, sometimes unexpected area. Trigonometric substitution is one of many substitution methods of integration where a function or expression in a given integral is replaced by trigonometric functions such as sin, cos, tan, etc. Integration by replacement is a good and easiest approach that anyone can make. It is used when we replace the function, whose derivative is already included in the given integral function. This simplifies the function and gives a function of simple integrals that we can easily integrate. It is also known as u-substitution or reverse chain rule Or in other words, using this method, we can easily evaluate integrals and antiderivatives.

Keywords: mathematics, trigonometric substitution, irrational equations, algebraic solution

INTRODUCTION

In mathematics, trigonometric substitution is the substitution of trigonometric functions for other expressions. In calculus, trigonometric substitution is a technique for evaluating integrals. Moreover, the trigonometric identity can be used to simplify certain integrals containing radical terms. As with other methods of integration by substitution, when evaluating a definite integral, it may be simpler to completely derive the antiderivative before applying the limits of integration.

Work on the use of trigonometric substitution for solving algebraic problems is best organized in optional mathematics classes. At the same time, it is advisable to offer students various problems to solve: rational and irrational equations, inequalities, their systems, tasks for finding the largest and smallest value of a function, tasks with parameters.

It is desirable to create a paper that would contain a selection of different algebraic problems solved by applying trigonometric substitution, not limited to considering a special class of problems.

The Purpose of the Work: to develop a methodology for using trigonometric substitution to solve algebraic problems for students of older grades in optional classes in classes with detailed study of mathematics.

Subject of Study: the process of applying trigonometric substitution as a method for solving various algebraic problems.

Subject of Research: organization of students' activities in mastering trigonometric substitution in elective classes in classes with advanced mathematics.

The study is based on the hypothesis that using a methodology developed on the basis of a comparative analysis of solving a large number of tasks will enable the development of students' creative abilities and preparation for entrance exams at serious universities.

METHODS OF SOLVING IRRATIONAL EQUATIONS

Irrational Equations

Irrational equations are often encountered in entrance exams in mathematics, because with their help it is easy to diagnose the knowledge of concepts such as equivalent transformations, domain of definition and others. Methods of solving irrational equations, as a rule, are based on the possibility of replacing (with the help of some transformations) an irrational equation with a rational one, which is either equivalent to the original irrational equation or is a consequence of it. Most often, both sides of the equation are raised to the same power. Equivalence is not violated when both parts are raised to an odd power. Otherwise, it is necessary to check the solutions found or evaluate the sign of both parts of the equation. But there are other tricks that can be more effective in solving irrational equations. For example, the method of trigonometric substitution.

Example 1: Solve the equation

$$\sqrt{\frac{1+2x\sqrt{1-x^2}}{2}}+2x^2=1$$

Solution Using Trigonometric Substitution

Because $1-x^2\geq 0$, that $|x|\leq 1$. Therefore, one can put $x=\sin\alpha,\quad \alpha\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$.

The equation will take the form

$$\sqrt{\frac{1+2\sin\alpha\cos\alpha}{2}}=1-2\sin^2\alpha \Leftrightarrow \frac{|\sin\alpha+\cos\alpha|}{\sqrt{2}}=\cos 2\alpha \Leftrightarrow \left|\sin\left(\alpha+\frac{\pi}{4}\right)\right|=\cos 2\alpha$$

Let's put $\alpha+\frac{\pi}{4}=u$ where $u\in\left[-\frac{\pi}{4};\frac{3\pi}{4}\right]$, then

$$|\sin u|=\sin 2u \Leftrightarrow \left[\begin{array}{l}\begin{cases}\sin u\rangle 0\\ \cos u=\frac{1}{2}\end{cases}\\ \sin u=0\\ \begin{cases}\sin u\langle 0\\ \cos u=-\frac{1}{2}\end{cases}\end{array}\right. \Leftrightarrow \left[\begin{array}{l}u_1=\frac{\pi}{3}\\ u_2=0\end{array}\right.$$

$$x_1 = \sin\left(u_1 - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$x_2 = \sin\left(u_2 - \frac{\pi}{4}\right) = \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

Answer: $\left\{-\frac{\sqrt{2}}{2}; \frac{\sqrt{6}-\sqrt{2}}{4}\right\}$.

Algebraic Solution

$$\sqrt{\frac{1+2x\sqrt{1-x^2}}{2}} + 2x^2 = 1 \Leftrightarrow \frac{1}{\sqrt{2}}\sqrt{\left(x+\sqrt{1-x^2}\right)^2} = 1-2x^2 \Leftrightarrow \frac{\left|x+\sqrt{1-x^2}\right|}{\sqrt{2}} = 1-2x^2$$

Because $1-2x^2 \geq 0$, that $1-x^2 \geq x^2$, $\sqrt{1-x^2} \geq |x|$. Means, $x+\sqrt{1-x^2} \geq 0$, so you can expand the module

$$\frac{x+\sqrt{1-x^2}}{\sqrt{2}} = 1-2x^2 \Leftrightarrow \frac{x+\sqrt{1-x^2}}{\sqrt{2}} = \left(1-x^2\right) - x^2 \Leftrightarrow \frac{x+\sqrt{1-x^2}}{\sqrt{2}} = \left(\sqrt{1-x^2}+x\right)\left(\sqrt{1-x^2}-x\right) \Leftrightarrow$$

$$\Leftrightarrow \left(\sqrt{1-x^2}+x\right)\left(\frac{1}{\sqrt{2}} - \left(\sqrt{1-x^2}-x\right)\right) = 0 \Leftrightarrow \left[\begin{array}{l}\sqrt{1-x^2}+x=0\\ \frac{1}{\sqrt{2}} = \sqrt{1-x^2}-x\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\begin{cases}2x^2=1\\ x\geq 0\end{cases}\\ \begin{cases}4x^2+2\sqrt{2}x-1=0\\ x\geq -\frac{\sqrt{2}}{2}\end{cases}\end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l}x = \frac{\sqrt{6}-\sqrt{2}}{4}\\ x = -\frac{\sqrt{2}}{2}\end{array}\right. .$$

Answer: $\left\{-\frac{\sqrt{2}}{2}; \frac{\sqrt{6}-\sqrt{2}}{4}\right\}$.

Solving an equation in an algebraic way requires a good skill in carrying out identical transformations and competent handling of equivalent transitions. But in general, both approaches are equivalent.

Algebraic Solution

Let's square both sides of the equation

$$x + \frac{x}{\sqrt{x^2-1}} = \frac{35}{12} \Leftrightarrow x^2 + \frac{x^2}{x^2-1} + \frac{2x^2}{\sqrt{x^2-1}} = \frac{1225}{144} \Leftrightarrow \frac{x^4}{x^2-1} + \frac{2x^2}{\sqrt{x^2-1}} = \frac{1225}{144}$$

We introduce the replacement $\frac{2x^2}{\sqrt{x^2-1}}=y, \quad y>0$, then the equation will be written in the form

$$y^2+2y-\frac{1225}{144}=0$$

$$\frac{D}{4}=1+\frac{1225}{144}=\frac{1369}{144}=\left(\frac{37}{12}\right)^2$$

$$\left[\begin{array}{l} y_1=\frac{25}{12} \\ y_2=-\frac{49}{12} \end{array}\right.$$

The second root is redundant, so consider the equation

$$\frac{x^2}{\sqrt{x^2-1}}=\frac{25}{12} \Leftrightarrow \frac{x^4}{x^2-1}=\frac{625}{144} \Rightarrow 144x^2-625x^2+625=0$$

$$D=625^2-4\cdot 625\cdot 144=625(625-576)=25^2\cdot 7^2=175^2$$

$$\left[\begin{array}{l} x^2=\frac{800}{288}=\frac{25}{9} \\ x^2=\frac{450}{288}=\frac{25}{16} \end{array}\right.$$

Because $x>0$, that

$$\left[\begin{array}{l} x_1=\frac{5}{3} \\ x_2=\frac{5}{4} \end{array}\right.$$

Answer:

$$\left\{\frac{5}{3};\frac{5}{4}\right\}.$$

In this case, the algebraic solution is technically simpler, but it is necessary to consider the above solution using a trigonometric substitution. This is due, firstly, to the non-standard nature of the substitution itself, which destroys the stereotype that the use of trigonometric substitution is possible only when $|x|\le 1$. It turns out if $|x|>1$ trigonometric substitution also finds application. Secondly, there is a certain difficulty in solving the trigonometric equation $\frac{\sin\alpha+\cos\alpha}{\sin\alpha\cos\alpha}=\frac{35}{12}$, which is reduced by introducing a change to a system of equations. In a certain sense, this replacement can also be considered non-standard, and familiarity with it allows you to enrich the arsenal of tricks and methods for solving trigonometric equations.

TRIGONOMETRICAL PROBLEMS OF INCREASED COMPLEXITY

Each educational trigonometric problem is solved on the basis learning activities that should be the subject of purposeful formation in the process of student activity in problem solving. To learn how to solve examples, equations and inequalities well any particular topic, it is necessary to carefully study the theoretical one's material, own formulas, tables, drawings.

Things related to trigonometric equations and inequalities, occupies a significant place in the school mathematics course and we are aware that these themes are widely used in certain sections mathematics, including solving applied problems. Often when solving trigonometric problems of increased complexity different solutions should be considered and which of them should be compared the most rational. One such method in mathematics is the use of trigonometric substitution in solving problems. He is one of the effective methods, especially in cases where needed to solve non-standard tasks.

Indeed, sometimes it is difficult to immediately guess which one substitution must be applied to resolve one or the other mathematical problem. To do this, you must master the basic methods of using trigonometric substitution, be able to analyze the conditions of the task, being able to assign the task to a specific type, etc. In this regard, before using the trigonometric method substitution, it is necessary to repeat the following topics with the students: properties of trigonometric functions; trigonometric formulas; methods solutions of rational, fractional-rational, irrational equations and inequalities, their systems; research and construction of function graphs; finding the largest and smallest value of a function; calculation integral etc.

CONCLUSION

The purpose of this paper is to study some integral problems of fractions. The main methods we used are the trigonometric methods of substitution and variable change for the fractional calculus based on Jumari's modification of the R-L fractional calculus. New multiplication plays an important role in this article. In fact, the results we obtained are natural generalizations of those in classical calculus. On the other hand, the new multiplication we have defined is a natural operation of fractional analytic functions. In the future, we will use these methods to solve problems in engineering mathematics and fractional differential equations.

Modern mathematics education, its harmonization with the requirements of scientific and technological development and progress, permanently requires the introduction of innovations that contribute to the modernization, rationalization and efficiency of the teaching process. In order to comprehensively understand and solve problems related to the teaching and learning of mathematics, mathematics education researchers pay significant attention to mathematical representations, visualization and modern educational technologies, their role and importance for the learning process. Multiple representations, visualization and educational technology, recognized as necessary components of mathematics education, need to be implemented in all segments of teaching, due to their potential to promote mathematical insight and understanding and improve the learning process. The mentioned aspects are particularly important for the study of functions, a fundamental concept of mathematics teaching.

In modern education, one of the important goals is to enable students to understand graphic representations of natural and social phenomena, which is considered a necessary competence in professional and everyday communication. However, regardless of the fact that competences in working with graphic representations are recognized as important outcomes of the teaching process, some experiences and research results (e.g., Leinhardt et al., 1990; Woolnough, 2000) show that many pupils/students have superficial and incomplete knowledge about this important area. The difficulties of some students arise from limited experience in working with graphic representations and their connection with symbolic representations. Such a situation most often arises as a consequence of the way in which the concept of function is processed in schools, when algebraic representations are the focus of learning, while graphic representations and visual approaches to learning are marginalized.

The introduction of materials related to trigonometric substitution in extracurricular classes in classes with a detailed study of mathematics contributes to the development of students' creative abilities and prepares them for university entrance exams with increased requirements for mathematics.

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