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An Exact Solution for Decay of Grid Produced-Turbulence

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Abstract:
A new approach to the exact solution for decay of grid-produced turbulence in the final period has been proposed. The governing equations are the two-point and three-point velocity correlation equations in which the quartic correlations are neglected as the closure assumption, and the pressure-velocity correlations are neglected tentatively. Without recourse to the isotropic conditions, these equations are found to be separable into a pair of Oseen (type) equations. As a result, the double- and triple-correlations are solved analytically as an initial value problem. The effect of the triple correlation adds a correction term proportional to x^{-4} to the well-known decay law $x^{-5/2}$ for the turbulent energy in the final period: $<(\Delta u)^2>=Ax^{-5/2}+ Bx^{-4}$, where $<(\Delta u)^2>$ is the turbulent energy, A and B are constants determined by the initial conditions, and x is the streamwise coordinate.

PART: 1
INTRODUCTION

"Turbulence" has been one of the most mysterious problems to scientists for several centuries. The nature of grid produced-turbulence has been an unsettled problem for almost one century. Partly for practical requirements (wind tunnel design) and partly for its theoretical tractability (homogeneous and isotropic turbulence) the research has been concentrated on grid-produced turbulence. However, it is impossible to over-emphasize the fact that mainly because of the nonlinearity of the Navier-Stokes equation, no-one has ever solved any problem about turbulence theoretically in concrete situations. Tsugé [3] has proposed to use the sequence of separated points velocity correlation equation, viz. the two-point velocity correlation equation, the three-point velocity correlation equation, and so on rather than conventional velocity correlation equation, say one-point velocity correlation equation [1,2]. Though Tsugé [3] has derived the two-point counterpart of the Navier-Stokes equation based on Klimontovich [4] formalism, those equations can be formally derived by using the Navier-Stokes equation: It should be noted that the formal derivation of his correlation equation is similar to Hinze's [5] two-point velocity correlation equation, which has been thought of a "entirely intractable" dueto the six-dimensional nature under general three-dimensional flow situations. Tsugé [3], however, has shown that these types of correlation equations [3, 5] are separable into a pair of Orr-Sommerfeld type equations at the respective points. In the present paper, it will be shown how the double correlations can be solved analytically with the aim of analyzing the unexplained phenomena of fully developed grid-produced turbulence. Using the experimental initial values in preference to the Loitsyanskii invariant [6], the double correlations will be solved analytically.

THE CORRELATION EQUATIONS

The two-point and the three-point velocity correlation equations will be formally derived for the general case of inhomogeneous and anisotropic turbulence.

The two-point velocity correlation equation has the form

$$<\Delta u_i(a) NS[\mathring{\mathbf{u}}(b), \mathring{p}(b)]_l + \Delta u_l(b) NS[\mathring{u}(a), \mathring{p}(a)]_i> = 0 \qquad (2.1)$$

with the following definition

$$NS(\mathbf{u}, p) \equiv (\partial/\partial t + u_j \cdot \partial/\partial x_j - \upsilon\nabla^2)\, u_i + 1/\rho \cdot \partial p/\partial x_i \qquad (2.2)$$

where bracket <> denotes an ensemble average, arguments (a) and (b) mean point A and point B, respectively, $\mathring{z}$ stands for instantaneous fluid dynamic quantity, z is its ensemble average, Δz is the fluctuation given by

$$\Delta z = \mathring{z} - z, \qquad (2.3)$$

and u_i Eulerian velocity, t time, x_j Eulerian Cartesian coordinates, ρ density, p static pressure, ∇^2 Laplacian operator, and υ kinematic viscosity. It may be worth noting here that Eq. (2.1) is similar to Hinze's two-point velocity correlation equation. The solenoidal conditions of the two-point velocity correlations are

$$\partial R^{(,1)}_{I,l}(a, b)/\partial x_i = \partial R^{(,1)}_{I,l}(a, b)/\partial x_l = 0, \qquad (2.4)$$

Where,

$$R^{(,1)}_{I,l}(a, b) = <\Delta u_i(a)\Delta u_l(b)>$$

is the two-point double velocity correlation.

THE APPLICATION TO GRID-PRODUCED TURBULENCE

As a matter of course, the turbulence produced by the grid mesh is not what is called isotropic. The former has a definite spatial directivity, viz. the direction of the main flow

$$u = (\dot{U}, 0, 0), \qquad (3.1)$$

while the latter has not, where $\dot{U}$ is the constant main flow velocity. Now it will be shown that the present method enables the solution for the double and triple correlations to be obtained without introducing the isotropic condition.

In the case of the present flow field, i.e., condition (3.1), eq (2.1) and (2.5) become, respectively,

$$\{\dot{U}[\partial/\partial x_1(a) + \partial/\partial x_1(b)] - \upsilon[\nabla^2(a) + \nabla^2(b)]\}\, R_{ij}^{(1,1)}(a, b)$$
$$= -\partial R_{ijr}^{(1,1,1)}(a, b, a)/\partial x_r(a) - \partial R_{ijr}^{(1,1,1)}(a, b, b)/\partial x_r(b), \qquad (3.2)$$

$$\{\dot{U}[\partial/\partial x_1(a) + \partial/\partial x_1(b) + \partial/\partial x_1(c)] - \upsilon[\nabla^2(a) + \nabla^2(b) + \nabla^2(c)]\}\, R_{ijr}^{(1,1,1)}(a, b, c) = 0, \qquad (3.3)$$

where the time derivative terms have been neglected because a time-dependent solution for fluctuation is not to be expected under the steady primary flow, and where the pressure-velocity correlations are also neglected; The pressure-velocity correlations were shown by Batcheor [7] to be identically zero for the case of homogeneous turbulence. For later convenience, the non-dimensional length x, double correlations R_{ij}, and triple correlations R_{ijr} are introduced by the following re-definition,

$$X=x/M, \tag{3.4}$$

$$R_{ij}=R_{ij}/\dot{U}^2, \tag{3.5}$$

$$R_{ijr}= R_{ijr}/\dot{U}^3, \tag{3.6}$$

and the Reynolds umber is defined as follows,

$$R=M\dot{U}/\upsilon, \tag{3.7}$$

where M is the mesh size of the grid. Then, the non-dimensional versions of eq s (3.2) and (3.3) ae simply obtainable by replacing in these equations,

$$\dot{U}=1,\ \upsilon=R^{-1}. \tag{3.8}$$

Therefore, the two equations become, respectively,

$$[\partial/\partial x_1(a)+ \partial/\partial x_1(b)]-1/R \cdot[\nabla^2(a)+ \nabla^2(b)]\}\ Rij^{(1,1)}(a, b)=-\partial R_{ijr}{}^{(1,1,1)}(a, b, a)/\partial x_r(a)-\partial R_{ijr}{}^{(1,1,1)}(a, b, b)/\partial x_r(b), \tag{3.9}$$

$$[\partial/\partial x_1(a)+ \partial/\partial x_1(b)+ \partial/\partial x_1(c)]-1/R \cdot[\nabla^2(a)+ \nabla^2(b)+ \nabla^2(c)]\}\ R_{ijr}{}^{(1,1,1)}(a, b, c)=0, \tag{3.10}$$

It is obvious that eq (3.10) is solvable by the method of variable disparation, viz.

$$R_{ijr}{}^{(1,1,1)}(a, b, c)=\varphi_i(a)\varphi_j(b)\varphi_r(c), \tag{3.11}$$

and φ_s follows the following equation

$$(\partial/\partial x_1-R^{-1}\nabla^2-i\lambda)\varphi_s=0, \tag{3.12}$$

where iλ is the separation constant such that the general solution is expressible in the form

$$R_{ijr}{}^{(1,1,1)}(a, b, c) = \int \varphi_i(a)\, \varphi_j(b)\varphi_r(c)\delta[\lambda(a)+ \lambda(b)+\lambda(c)]d\lambda(a)\, \lambda(b)\, \lambda(c), \tag{3.13}$$

where δ is the Dirac delta function.

It is easily seen that eq (3.12) corresponds to the special case of the Oseen equation for waves travelling in a uniform flow with frequency λ. Such waves decay due to viscous effects and dispersion. This fact immediately suggests that a solution of the following form is sought,

$$\varphi_s=\int A_s(\mathbf{k}, \beta, \lambda)\exp(-\beta x_1+ik_ix_i)dk_2dk_3,\ (\beta -ik_1)^2+R(\beta -ik_1)+i\lambda R-k_2{}^2-k_3{}^2=0, \tag{3.15}$$

which assures that φ_s is the solution of eq (3.12). After eq (3.15) is decomposed into the real and the imaginary parts, respectively, β and λ become as the first approximation,

$$\beta \cong k^2/R, \tag{3.16}$$

$$\lambda \equiv k_1, \tag{3.17}$$

where $k^2=k_1^2+k_2^2+k_3^2$. If the expression like eq (3.14) for points a, b, and c, respectively, are substituted into eq (3.13), we obtain the general solution for the triple correlations.

$$R_{ijr}^{(1,1,1)}(a, b, c) = \int C_{ijr}\exp\{-1/R[k^2(a)x_1(a)+k^2(b)x_1(b)+k^2(c)x_1(c)]+i[k_i(a)x_i(a)+ k_i(b)x_i(b)+ k_i(c)x_i(c)\ _i(a)]\}\,\delta[\mathbf{k}(a)+\mathbf{k}(b)+k_i(c)]dk(a)\ k(b)\ k(c), \tag{3.18}$$

where we put $C_{ijr}=A_i A_j A_r$, and we use the relations (3.16) and (317). Furthermore, we use the condition that the triple correlations are homogeneous in planes parallel with the grid. Once we can determine the C_{ijr} from the initial conditions, we will be able to solve $R_{ij}^{(1,1)}(a, b)$ in eq (3.9) formally,

$$R_{ij}^{(1,1)}(a, b)= [R_{ij}^{(1,1)}(a, b)]_c+ [R_{ij}^{(1,1)}(a, b)]_p , \tag{3.19}$$

Where $[R_{ij}^{(1,1)}(a, b)]_c$ and $[R_{ij}^{(1,1)}(a, b)]_p$ are the complementary and the particular solutions, respectively.

THE SOLUTION N THE FINAL PERIOD OF DECAY

As is well known, the complementary solution is such a solution that can be obtained by putting the right-hand side of eq (3.9) to zero, viz.

$$[\partial/\partial x_1(a)+ \partial/\partial x_1(b)]-1/R\cdot[\nabla^2(a)+ \nabla^2(b)]\}\ [R_{ij}^{(1,1)}(a, b)]_c=0\ . \tag{4.1}$$

Similarly to the solution for $R_{ijr}^{(1,1,1)}(a, b, c)$ in eq (3.10), the complementary solution is solvable by the method of variable-separation in the following manner. The solution is expressible as follows,

$$[R_{ij}^{(1,1)}(a, b)]_c=\int \varphi_i(a)\varphi_j(b)\delta[\lambda(a)+ \lambda(b)]\ d\lambda(a)\ \lambda(b). \tag{4.2}$$

Moreover, φ_s has a similar form φ_s, i.e.

$$\varphi_s=\int A_s(k, \beta, \lambda)\exp(-\beta x_1+ik_ix_i)dk_2dk_3, \tag{4.3}$$

where β ad λ satisfy the same dispersion relation as eq (3.15). Hence, substituting the expression like eq (4.3) for points a and b into eq (4.2), we obtain the complementary solution,

$$[R_{ij}^{(1,1)}(a, b)]_c= \int C_{ij}\exp\{-1/R\cdot[k^2(a)x_1(a)+ k^2(b)x_1(b)] + i[k_i(a)x_i(a)+k_i(b)x_i(b)]\}\delta[\mathbf{k}(a)+\mathbf{k}(b)]d\mathbf{k}(a)d\mathbf{k}(b), \tag{4.4}$$

where we put $C_{ij}=A_i A_j$, and we use the relations (3.16) and (3.17) as well as the condition that the double correlations are homogeneous in the planes parallel with the grid.

For later convenience, we will rewrite eq. (4.4) as follows

$$[R_{ij}^{(1,1)}(a, b)]_c = \int C_{ij} \exp[-2k^2/R(x-x_o) + ik_i r_i] dk, \tag{4.5}$$

where $x - x_o = [x_1(a) + x_1(b)]$, **r** = **x**(a) – x(b), and x_o is a position behind the grid, where the longitudinal double velocity correlation has a Gaussian distribution: This fact is supported by many experiments, viz. Batchelor-Townsend (1948) (see Fig.3), Stewart (1951), and Van Atta Chen (1969). Using the longitudinal double velocity correlation measure by Batchelor-Townsend (1948) as an initial condition, the unknown constant C_{ij} can be determined as follows. Substituting the double velocity correlations at the point $x = x_o$ in eq. (4.5), we get

$$(f-g)\, r_i r_j / r^2 + g\delta_{ij} = \int C_{ij} \cdot \exp(ik_i r_i) dk. \tag{4.6}$$

Moreover, operating

$$1/(2\pi)^3 \int exp(-ik'_i \cdot r_i) dr$$

to the both sides of eq. (4.6), we obtain

$$1/(2\pi)^3 \int [(f-g)\, r_i r_j / r^2 + g\delta_{ij}] \exp(-ik'_i \cdot r_i) dr = C_{ij}, \tag{4.7}$$

where we use the definition of delta function, that is

$$1/(2\pi)^3 \int exp\, ir \cdot (k-k')\, dr dk = \int \delta[k-k'] \cdot C_{ij} \cdot dk = C_{ij}$$

Before, we integrate the l.h.s. of eq. (4.7), we must notice the following facts. In order to integrate the l.h.s., we must use such a polar coordinate that the direction of k′ coincides with Z-axis; the direction of k′ always coincides with that of $r^*_3 (= Z^*)$. Therefore, the direction of X^* is generally different from the stream-wise coordinate X (Figure 1).

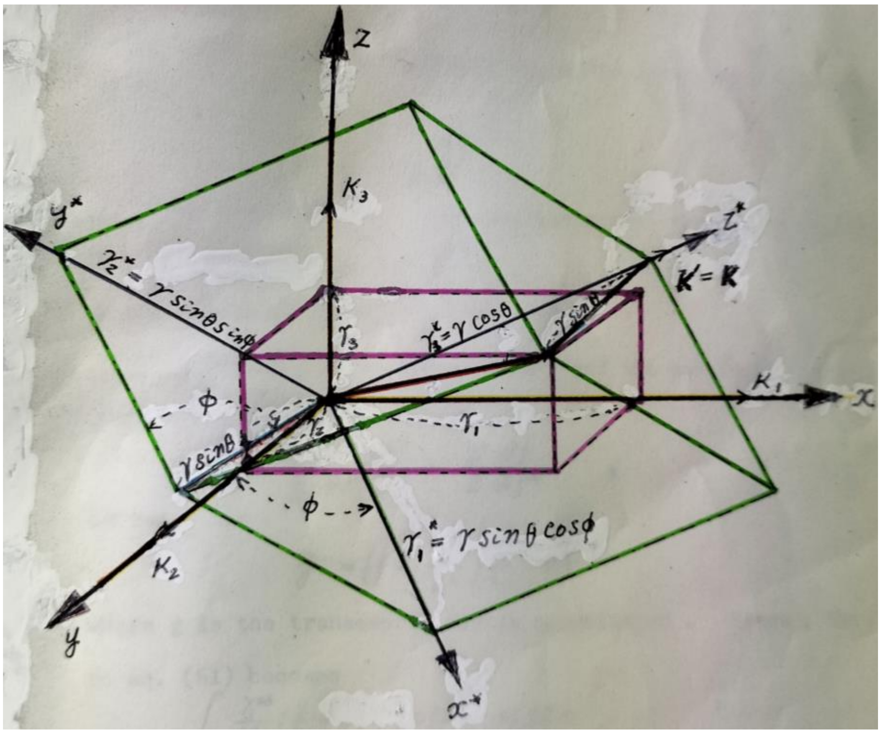

Figure 1: Definition sketch of coordinates.

Whereas we usually adopt the coordinate X as a reference axis in determining the double velocity correlations. In order to unify these two coordinate systems. We introduce next notation.

$$r_i=\hat{i}\hat{l}r^*_l, \tag{4.8}$$

where $\hat{i}\hat{l}$ is direction cosine between the two coordinate systems. For example, we will calculate the following term.

$$1/(2\pi)^3\int(r_1r_1/r^2)\,(f-g)\cdot\exp(-ik_i\cdot r_i)\,dr$$

$$=1/(2\pi)^3\int(\hat{1}\hat{l}\hat{1}\hat{m}r^*_lr^*_m/r^2)\,(f-g)\cdot\exp(-ik_i\cdot r_i)\,dr,$$

$$=1/(2\pi)^3\int(\hat{1}\hat{1}\hat{1}\hat{1}r^{*2}_1+\hat{1}\hat{2}\hat{1}\hat{2}r^{*2}_2+\hat{1}\hat{3}\hat{1}\hat{3}r^{*2}_3)/r^2\cdot(f-g)\cdot\exp(-ik_i\cdot r_i)dr,$$

$$=1/(2\pi)^3\int(\hat{1}\hat{1}\hat{1}\hat{1}r^{*2}_1+\hat{1}\hat{2}\hat{1}\hat{2}r^{*2}_2+\hat{1}\hat{3}\hat{1}\hat{3}r^{*2}_1-\hat{1}\hat{3}\hat{1}\hat{3}r^{*2}_1+\hat{1}\hat{3}\hat{1}\hat{3}r^{*2}_3)/r^2\cdot(f-g)\cdot\exp(-ik_i\cdot r_i)dr,$$

$$=1/(2\pi)^3\int[(\hat{1}\hat{1}\hat{1}\hat{1}r^{*2}_1+\hat{1}\hat{2}\hat{1}\hat{2}r^{*2}_2+\hat{1}\hat{3}\hat{1}\hat{3}r^{*2}_1)+\hat{1}\hat{3}\hat{1}\hat{3}\,(r^{*2}_3-r^{*2}_1)]/r^2\cdot(f-g)\cdot\exp(-ik_i\cdot r_i)dr,$$

$$=1/(2\pi)^3\int[(\hat{1}\hat{1}\hat{1}\hat{1}r^{*2}_1+\hat{1}\hat{2}\hat{1}\hat{2}r^{*2}_1+\hat{1}\hat{3}\hat{1}\hat{3}r^{*2}_1)+\hat{1}\hat{3}\hat{1}\hat{3}\,(r^{*2}_3-r^{*2}_1)]/r^2\cdot(f-g)\cdot\exp(-ik_i\cdot r_i)dr,$$

$$=1/(2\pi)^3\int[\hat{1}\hat{i}\hat{1}\hat{i}r^{*2}_1+\hat{1}\hat{3}\hat{1}\hat{3}\,(r^{*2}_3-r^{*2}_1)]/r^2\cdot(f-g)\cdot\exp(-ik_i\cdot r_i)dr,$$

$$=1/(2\pi)^3\int[\hat{1}\hat{i}\hat{1}\hat{i}r^{*2}_1+\hat{1}\hat{3}\hat{1}\hat{3}\,(r^{*2}_3-r^{*2}_1)]/r^{*2}\cdot(f-g)\cdot\exp(-ikr^*)d\mathbf{r}^*, \tag{4-9}$$

where we use the relations,

$$k'=k,$$

$$1/(2\pi)^3\int(\hat{1}\hat{1}\hat{1}\hat{2}r_1^*r_2^*)\,(f-g)\cdot\exp(-ik_i\cdot r_i)\,d\mathbf{r}=0,\ \ldots\ldots,$$

$$1/(2\pi)^3\int(r^{*2}_1/r^2)\,(f-g)\cdot\exp(-ik_i\cdot r_i)\,dr=1/(2\pi)^3\int(r^{*2}_2/r^2)\,(f-g)\cdot\exp(-ik_i\cdot r_i)\,d\mathbf{r}.$$

At this stage, we introduce Batchelor-Townsend (1948)'s longitudinal double velocity correlation,

$$f=\exp[-r^{*2}/(a^2M^2)], \tag{4-10}$$

where a is a constant, to be determined by the experiment. The normalized version of eq. (4.10) becomes

$$f=\exp(-r^{*2}/a^2), \tag{4-11}$$

where we use $r^*\rightarrow Mr^*$ for the normalization.

Furthermore, using the functional form f and the continuity relation of the double correlation,

$$g=f+r^*/2\cdot\partial f/\partial r^*,$$

we have

$$g=(1-r^{*2}/a^2)\ \exp(-r^{*2}/a^2), \qquad (4\text{-}12)$$

where g is the transverse double correlation. Hence, the following term in eq. (4.9) becomes

$$\int r_1^{*2}/r^{*2}(f-g)\ \exp(-ikr^*)\ dr^*, =\pi \int_0^\infty r^{*2}[r^{*2}/a^2\cdot\exp(-r^{*2}/a^2)\ [4\sin kr^*/(k^3r^{*3})-4\cos kr^*/(k^2r^{*2})]\ dr^*, =\pi^{3/2}/2\cdot a^3\exp(-a^2k^2/4). \qquad (4\text{-}13)$$

Similarly, to the above,

$$\int r_3^{*2}/r^{*2}(f-g)\ \exp(-ikr^*)\ dr^*, =\pi \int_0^\infty r^{*2}[r^{*2}/a^2\cdot\exp(-r^{*2}/a^2)\ [4\sin kr^*/(kr^*)+4\cos kr^*/(k^2r^{*2})-,4\cos kr^*/(k^3r^{*3})]\ dr^*, = \pi^{3/2}/2\cdot a^3\exp(-a^2k^2/4)- \pi^{3/2}/4\cdot a^5k^2\exp(-a^2k^2/4). \qquad (4\text{-}14)$$

Substituting eq. (4-13) and (4-14) in eq. (4.9), we get

$$1/(2\pi)^3\int r_1r_1/r^{*2}(f-g)\ \exp(-ikr^*)\ d\mathbf{r}^*=a^3/(16\pi^{3/2})\cdot\exp(-a^2k^2/4)\ (1-a^2k_1^2). \qquad (4\text{-}15)$$

In the same way, we can obtain the following relations,

$$1/\ (2\pi)^3 \int r_1r_2/r^{*2}(f-g)\ \exp(-ikr^*)d\mathbf{r}^*=a^3/(16\pi^{3/2})\cdot\exp(-a^2k^2/4)(-a^2k_1k_2/2) \qquad (4\text{-}16)$$

And

$$1/\ (2\pi)^3 \int r_ir_i/r^{*2}(f-g)\ \exp(-ikr^*)\ d\mathbf{r}^*=a^3/(16\pi^{3/2})\cdot\exp(-a^2k^2/4)(3 -a^2k^2/2). \qquad (4\text{-}17)$$

Therefore, from eq. (4-15), (4-16) and (4-17), we can get the general expression, viz.

$$1/\ (2\pi)^3 \int r_ir_j/r^{*2}(f-g)\ \exp(-ikr^*)d\mathbf{r}^*= A(\delta_{ij}-a^2k_ik_j/2), \qquad (4\text{-}18)$$

Where,

$$A=a^3/(16\pi^{3/2})\cdot\exp(-a^2k^2/4).$$

Similarly, to the above, the term in eq. (4.7) becomes

$$\delta_{ij}/(2\pi)^3\int g\ \cdot\exp(-ikr^*)\ d\mathbf{r}^*= \delta_{ij}\,/\ (2\pi)^3 \int(1-r^{*2}/a^2)\ \exp\ (-r^{*2}/a^2)\cdot \exp(-ikr*)d\mathbf{r}*=-A\delta_{ij} + A/2\cdot k^2a^2\delta_{ij}. \qquad (4\text{-}19)$$

Substituting eq. (4-18) and (4.19) in eq. (4-7), we finally have

$$C_{ij}=Aa^2k^2/2\cdot(\delta_{ij} - k_ik_j/k^2). \qquad (4\text{-}20)$$

Then, substituting C_{ij} in the above that are determined with the initial condition in eq. (4-5), we can obtain the turbulent energy decay law in the final period. Namely,

$$[R_{ij}^{(1,1)}\,(a,\,b)]_c=\pi^{1/2}a^5/32\int(k^2\delta_{ij}-k_ik_j)\ \exp(-a^2k^2/4)\cdot\exp[-2k^2/R\cdot(x-x_0)]+ikr^*]d\mathbf{k}. \qquad (4\text{-}21)$$

By the definition of the turbulent energy, we put the subscripts i=j and $r^*=0$ in eq. (4-21), the energy decay law becomes

$$\bar{u}^2_c = a^5/(32\pi^{3/2})\int(k^2\delta_{ij}-k_ik_j)\exp(-a^2k^2/4)\cdot\exp[-2k^2/R\cdot(x-x_0)]d\mathbf{k} = 3a^2/32\cdot[a^2/4 + 2/R\cdot(x-x_0)]^{-5/2}, \quad (4\text{-}22)$$

where $\bar{u}^2_c$ is turbulent energy in the final period of decay. It may be worth noting that x_0 is the position where the longitudinal double velocity correlation possesses a Gaussian distribution. Now in Fig.2, it is clearly seen that turbulent energy in the final period decays according to $x^{-5/2}$. Owing to this reason, assume that the Gaussian distribution is realized at the position, viz.
$x_0 = Ra^2/8$.

In this particular case, the energy decay law in the final period can be expressed by

$$\bar{u}^2_c = 3a^2/32\cdot(R/2)^{5/2}\cdot x^{-5/2}. \quad (4.23)$$

Let us change the variable x in eq (4.23) to the timer t by introducing the so-called Taylor's hypothesis, t=x/U, the present decay law agrees with the existing theories in the final period of decay such as Loitsyyanskii [12], Batchelor [6], and Deissler [13]. However, Phillips [14] and Saffman [15] lead -3/2 power law in the final period theoretically.

The present decay law is also in good agreement with Batchelor & Townsend's by an experiment; both of these experiments show that turbulent energy decreases inversely as the square of decay time. Referring to eq (4.22), if the Reynolds number R is small compared with the distance $(x-x_0)$, the turbulent energy decays as $(x-x_0)^{-5/2}$. In fact, the previous experiments have been always performed under low Reynolds number, i.e., $R \leq 1000$. Thus, it is necessary to establish the effect of Reynolds number by conducting experiments at high Reynolds number. Based upon eq (4.22), we can suggest in case of high Reynolds number the initial stage of grid-produced turbulence must persist over a greater distance than in the case of low Reynolds number. In contrast to the final period of decay (e.g. [8], [12]), most experimental turbulence energy decays as t^{-1} (or x^{-1}) (e.g. [8],[13]). Therefore, the present theory conjectures that in the classical sense the initial period might be able to exist even more downstream of the grid under high Reynolds number.

In fact, when $8/(a^2R)\cdot(x-x_0) \ll 1$, eq (4.22) can be expressed by

$$\bar{u}^2_c = 3(2)^{1/2}/a^3[1-20/(a^2R)\cdot(x-x_0)]. \quad (4.24)$$

The above relation indicates that the turbulence energy decays as x^{-1} essentially. In other words, we may not need the triple correlations in order to obtain the turbulent energy decay law covering the whole region behind the grid. It may be useful to know that the turbulent energy decay law behind the gris is an initial value problem mathematically

DISCUSSIONS

In support of this proposition, Uberoi & Wallis [14] examined the effects of the grid geometry. Also, Ling & Wan [15] made an experimental study of weak turbulence created by a mechanically agitated grid, and found that turbulent energy decay law depends on the velocity ratio of agitator to the mean flow.

There are two significant limitations in the present theory. Firstly, the pressure-velocity correlations are neglected, and secondly the initial conditions for the triple correlations are not given due to the lack of the experimental data. The former assumption, i.e., the pressure-velocity correlations are zero, has been shown to be valid only for the case of homogeneous turbulence [7], while the present theory only enquires homogeneity in planes parallel with the grid. It should be here noted that the present theory treats much more general flow field than the classical one, e.g. [1],[2] and [7].

Finally, the some works for including the pressure-velocity correlations into the present theory and for applying the present theory to oceanic turbulence have been conducted by Tsugé [16] and Nakagawa [17], respectively. For example, oceanic turbulence in the upper layers of the ocean, the turbulence is approximately homogeneous in horizontal planes [18], so that the present theory may be applicable easily. It is believed that the present theoretical approach to turbulence has great possibility in its applicability for various turbulence problems.

CONCLUDING REMARKS

Without using homogeneous and isotropic conditions, the double correlations are exactly solved as an initial value problem. The solution is expressed by

$$\bar{u}^2{}_c = 3a^2/32 \cdot (R/2)^{5/2} \cdot x^{-5/2},$$

where a is a constant to be determined by the measurement of the double velocity correlation at the initial point behind the grid, R is the Reynolds number defined by UM/υ, U is the flow velocity in the upstream of the grid, M is the mesh size, υ is the kinematic viscosity of the fluid, and x is the distance from the grid in the downstream. It has been, however, assumed that the flow field behind the grid is homogeneous in the planes parallel with the grid together with the pressure-velocity correlations have been neglected tentatively.

In part 2, it will be challenged to solve the triple correlations in the same way as the double correlations. Such work has, therefore, been left for this part.

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PART 2

This paper is concerned with an exact solution of grid-produced turbulence in the transitional period of decay. This is part 2 of our previous paper entitled an exact solution of grid-produced turbulence: part 1, in which the turbulent intensity $\bar{u}^{2}$ behind the grid is expressed as follows,

$$\bar{u}^2=3a^2/32\cdot(R/2)^{5/2}\cdot x^{-5/2},$$

where a is a constant to be determined by the measurement of the double velocity correlation at the initial point behind the grid, R is the Reynolds number defined by UM/υ, U is the flow velocity in the upstream of the grid, M is the mesh size, υ is the kinematic viscosity of the fluid, and x is the distance from the grid taking positive in the downstream. That is, because the present part is an extension of the part1, both of the introduction are references are common, so the mathematical argument to derive the correction term will be started immediately. It is found that the inclusion of the triple velocity correlations adds a correct term $\sim x^{-4}$ to the forgoing solution $\sim x^{-52}$. The comparison of the theory and experiment has revealed that the agreement is satisfactory, and the correction term contributes to improve the degree of the agreement significantly.

THEORETICAL ANALYSES FOR THE GRID-PRODUCED TURBULEN IN THE TRANSITIONAL PERIOD OF DECAY

The two-point and the three-point velocity correlation equations will be formally derived for the general case of inhomogeneous and anisotropic turbulence (Tsugé 1974). The two-point velocity correlation equation has the form

$$<\Delta u_i(a)NS[\mathring{u}(b), \mathring{p}(b)]_l + \Delta u_l(b)NS[\mathring{u}(a), \mathring{p}(a)]_i>=0 \quad (1)$$

with the following definition

$$NS(\mathbf{u}, p)\equiv (\partial/\partial t + u_j\cdot \partial/\partial x_j - \upsilon\nabla^2)\, u_i + 1/\rho \cdot \partial p/\partial x_i, \quad (2)$$

where bracket <> denotes an ensemble average, arguments (a) and (b) mean point A and point B, respectively, $\mathring{z}$ stands for instantaneous fluid dynamic quantity, z is its ensemble average, Δz is the fluctuation given by

$$\Delta z=\mathring{z} - z, \quad (3)$$

and u_i Eulerian velocity, t time, x_j Eulerian Cartesian coordinates, ρ density, p static pressure, ∇^2 Laplacian operator, and υ kinematic viscosity. It may be worth noting here that (1) is similar to Hinze's two-point velocity correlation equation. The solenoidal conditions of the two-point velocity correlation are

$$\partial R^{(,1)}{}_{I,l}(a, b)/\partial x_i = \partial R^{(,1)}{}_{I,l}(a, b)/\partial x_l = 0, \quad (4)$$

Where,

$$R^{(,1)}{}_{I,l}(a, b)=<\Delta u_i(a)\Delta u_l(b)>$$

is the two-point double velocity correlation.

THE APPLICATION TO GRID-PRODUCED TURBULENCE

As a matter of course, the turbulence produced by the grid mesh is not what is called isotropic. The former has a definite spatial directivity, viz. the direction of the main flow

$$\upsilon = (\dot{U}, 0, 0), \tag{5}$$

while the latter has not, where $\dot{U}$ is the constant main flow velocity. Now it will be shown that the present method enables the solution for the double and triple correlations to be obtained without introducing the isotropic condition.

In the case of the present flow field, i.e., condition (5), eq (1) and (2) become, respectively,

$$\{\dot{U}[\partial/\partial x_1(a)+ \partial/\partial x_1(b)]-\upsilon[\nabla^2(a)+ \nabla^2(b)]\} R_{ij}^{(1,1)}(a, b)=$$

$$=-\partial R_{ijr}^{(1,1,1)} (a, b, a)/\partial x_r(a)-\partial R_{ijr}^{(1,1,1)}(a, b, b)/\partial x_r(b), \tag{6}$$

$$\{\dot{U}[\partial/\partial x_1(a)+ \partial/\partial x_1(b)+ \partial/\partial x_1(c)]-\upsilon[\nabla^2(a)+ \nabla^2(b)+ \nabla^2(c)]\} R_{ijr}^{(1,1,1)}(a, b, c)=0, \tag{7}$$

where the time derivative terms have been neglected because a time-dependent solution for fluctuation is not to be expected under the steady primary flow, and where the pressure-velocity correlations are also neglected; The pressure-velocity correlations were shown by Batcheor [7] to be identically zero for the case of homogeneous turbulence. For later convenience, the non-dimensional length x, double correlations R_{ij}, and triple correlations R_{ijr} are introduced by the following re-definition,

$$X=x/M, \tag{8}$$

$$R_{ij}=R_{ij}/\dot{U}^2, \tag{9}$$

$$R_{ijr}= R_{ijr}/\dot{U}^3, \tag{10}$$

and the Reynolds umber is defined as follows,

$$R=M\dot{U}/\upsilon, \tag{11}$$

where M is the mesh size of the grid. Then, the non-dimensional versions of (1) and (2) are simply obtainable by replacing in these equations,

$$\dot{U}=1,\ \upsilon=R^{-1}. \tag{12}$$

Therefore, the normalized two equations become, respectively,

$$[\partial/\partial x_1(a)+ \partial/\partial x_1(b)]-1/R\cdot[\nabla^2(a)+ \nabla^2(b)]\} R_{ij}^{(1,1)} (a, b)=$$

$$=-\partial R_{ijr}^{(1,1,1)} (a, b, a)/\partial x_r(a)-\partial R_{ijr}^{(1,1,1)}(a, b, b)/\partial x_r(b), \tag{13}$$

$$[\partial/\partial x_1(a)+\partial/\partial x_1(b)+\partial/\partial x_1(c)]-1/R\cdot[\nabla^2(a)+\nabla^2(b)+\nabla^2(c)]\}\, R_{ijr}^{(1,1,1)}(a, b, c)=0, \quad (14)$$

It may be obvious that (11) is solvable by the method of variable separation, viz.

$$R_{ijr}^{(1,1,1)}(a, b, c)=\varphi_i(a)\varphi_j(b)\varphi_r(c), \quad (15)$$

and φ_s follows the following equation

$$(\partial/\partial x_1-R^{-1}\nabla^2-i\lambda)\varphi_s=0, \quad (16)$$

where iλ is the separation constant such that the general solution is expressible in the form

$$R_{ijr}^{(1,1,1)}(a, b, c) = \int \varphi_i(a)\,\varphi_j(b)\varphi_r(c)\delta[\lambda(a)+\lambda(b)+\lambda(c)]d\lambda(a)\,\lambda(b)\,\lambda(c), \quad (17)$$

where δ is the Dirac delta function.

It is easily seen that (16) is nothing more than a special case of the Oseen equation for waves travelling in a uniform flow with frequency λ. Such waves decay due to viscous effects and dispersion. This fact immediately suggests that a solution of the following form is sought,

$$\varphi_s=\int A_s(k, \beta, \lambda)\exp(-\beta x_1+ik_ix_i)dk_2dk_3, \quad (18)$$

$$(\beta-ik_1)^2+R(\beta-ik_1)+i\lambda R-k_2^2-k_3^2=0, \quad (19)$$

which assures that φ_s is the solution of (16). After (19) is decomposed into the real and the imaginary parts, respectively, β and λ become as the first approximation,

$$\beta\cong k^2/R, \quad (20)$$

$$\lambda\equiv k_1, \quad (21)$$

where $k^2=k_1^2+k_2^2+k_3^2$. If the expression like (18) for points a, b, and c, respectively, are substituted into (17), we obtain the general solution for the triple correlations.

$$R_{ijr}^{(1,1,1)}(a, b, c) =\int C_{ijr}\exp\{-1/R[k^2(a)x_1(a)+k^2(b)x_1(b)+k^2(c)x_1(c)]+i[k_i(a)x_i(a)+k_i(b)x_i(b)+k_i(c)x_i(c)\,{}_i(a)]\}\,\delta[k(a)+k(b)+k_i(c)]dk(a)\,k(b)\,k(c), \quad (22)$$

where we put $C_{ijr}=A_i A_j A_r$, and we use the relations (20) and (21). Furthermore, we use the

$$R_{ij}^{(1,1)}(a, b)=[R_{ij}^{(1,1)}(a, b)]_c+[R_{ij}^{(1,1)}(a, b)]_p, \quad (23)$$

where $[R_{ij}^{(1,1)}(a, b)]_c$ and $[R_{ij}^{(1,1)}(a, b)]_p$ are the complementary and the particular solutions, respectively.

THE SOLUTION IN THE TRANSITIONAL PERIOD OF DECAY

As we have already derived the complementary solution in (13) in the part 1, it is sufficient to solve the particular solution to obtain the one in the transitional period of decay.

$$[\partial/\partial x_1(a)+\partial/\partial x_1(b)]-1/R\cdot[\nabla^2(a)+\nabla^2(b)]\}R_{ij}^{(1,1)}(a,b)=-\partial R_{ijr}^{(1,1,1)}(a,b,a)/\partial x_r(a)-\partial R_{ijr}^{(1,1,1)}(a,b,b)/\partial x_r(b)=-\int C_{ijr}[-\beta(a)\delta_{r1}-\beta(-a-b)\delta_{r1}-ik_r(b)]\exp\{-[\beta(a)+\beta(-a-b)]x_1(a)-\beta(b)x_1(b)+ik(b)[-x(a)+x(b)]dk(a)dk(b)-\int C_{ijr}[-\beta(b)\delta_{r1}-\beta(-a-b)\delta_{r1}-ik_r(a)]\exp\{-[\beta(b)+\beta(-a-b)]x_1(b)-\beta(a)x_1(a)+ik(a)[x(a)-x(b)]dk(a)dk(b), \quad (24)$$

where we use (20). Comparing with the both sides of (24), the particular solution must have the form as follows,

$$[R_{ij}^{(1,1)}(a,b)]_p=\int D_{ij}\exp\{-[\beta(a)+\beta(-a-b)]x_1(a)-\beta(b)x_1(b)+ik(b)[-\mathbf{x}(a)+\mathbf{x}(b)]d\mathbf{k}(a)d\mathbf{k}(b)-\int E_{ij}\exp\{-[\beta(b)+\beta(-a-b)]x_1(b)-\beta(a)x_1(a)+ik(a)[\mathbf{x}(a)-\mathbf{x}(b)]d\mathbf{k}(a)d\mathbf{k}(b). \quad (25)$$

Moreover, substituting (25) in (24), we obtain the expression for the l.h.s. as

$$\int D_{ij}【-\beta(a)-\beta(-a-b)-\beta(b)-1/R\{[-\beta(a)-\beta(-a-b)-ik_1(b)]^2-2k_2^2(b)-2k_3^2(b)+[-\beta(b)+ik_1(b)]^2\}】\exp\{-[\beta(a)+\beta(-a-b)]x_1(a)-\beta(b)x_1(b)+ik(b)[-\mathbf{x}(a)+\mathbf{x}(b)]\}d\mathbf{k}(a)d\mathbf{k}(b)-\int E_{ij}【-\beta(a)-\beta(-a-b)-\beta(b)-1/R\{[-\beta(b)-\beta(-a-b)-ik_1(a)]^2-2k_2^2(a)-2k_3^2(b)+[-\beta(a)+ik_1(a)]^2\}】\exp\{-[\beta(b)+\beta(-a-b)]x_1(b)-\beta(a)x_1(a)+ik(a)[\mathbf{x}(a)-\mathbf{x}(b)]\}d\mathbf{k}(a)d\mathbf{k}(b). \quad (26)$$

Then, compare the r.h.s. of (24) with (26), we get the following two relations,

$$D_{ij}【-\beta(a)-\beta(-a-b)-\beta(b)-1/R\{[-\beta(a)-\beta(-a-b)-ik_1(b)]^2-2k_2^2(b)-2k_3^2(b)+[-\beta(b)+ik_1(b)]^2\}=-C_{ijr}[-\beta(a)\delta_{r1}-\beta(-a-b)\delta_{r1}-ik_r(b)], \quad (27)$$

And

$$E_{ij}【-\beta(a)-\beta(-a-b)-\beta(b)-1/R\{[-\beta(b)-\beta(-a-b)-ik_1(a)]^2-2k_2^2(a)-2k_3^2(a)+[-\beta(a)+ik_1(a)]^2\}】=-C_{ijr}[-\beta(b)\delta_{r1}-\beta(-a-b)\delta_{r1}-ik_r(a)]. \quad (28)$$

It is, therefore, evident if the three-point triple correlations, in which each of three points is definitely separated one another, are given by the measurement, it is possible to obtain the particular solution firmly; such a solution does not contain any undetermined constant in it.

It is the problem to specify the value of the constant C_{ijr} in (27) and (28) by using the three-point correlations within a plane behind the grid, where the plane is parallel with it. The three-point correlations play a role in the initial conditions in this mathematical problem. The particular solution (25) will be able to be derived once D_{ij} and E_{ij} are obtained as the function of wave numbers in principle.

Let's demonstrate how to obtain the triple correlation mathematically. Firstly, it is necessary for us to make the measurement to get an analytical expression of the three-point triple correlations at a position within a plane of $x_1(a)=x_1(b)=x_1(c)=0$ behind the grid, which is parallel to the plane. In this case, the three-point triple correlation

$$R_{ijr}^{(1,1,1)}(a,b,c)=\int C_{ijr}\exp\{-1/R[k^2(a)x_1(a)+k^2(b)x_1(b)+k^2(c)x_1(c)]+i[k_i(a)x_i(a)+k_i(b)x_i(b)+k_i(c)x_i(c)_i]\}\delta[\mathbf{k}(a)+\mathbf{k}(b)+k_i(c)]dk(a)\,k(b)\,k(c),$$

Becomes

$$[R_{ijr}^{(1,1,1)}(a, b, c)]_{x1(a)=x1(b)=x1(c)=0} = \int C_{ijr} \exp[i[k_i(a)x_i(a)+k_i(b)x_i(b)- k_i(a)x_i(c) -k_i(b)x_i(c)]\} d\mathbf{k}(a)\, \mathbf{k}(b)$$
$$= \int C_{ijr} \exp\{i[k_i(a)\xi_i+k_i(b)\eta_i]\} d\mathbf{k}(a)\, \mathbf{k}(b), \quad (29)$$

where,

$$\boldsymbol{\xi}=\mathbf{x}(a)-\mathbf{x}(c) \text{ and } \boldsymbol{\eta}=\mathbf{x}(b)-\mathbf{x}(c).$$

Then, operating

$1/(2\pi)^3\int e\exp[-i\xi_i k_i'(a)]\, d\boldsymbol{\xi}$ to the both sides of (29), we get

$$1/(2\pi)^3 \int [R_{ijr}^{(1,1,1)}(a,b,c)]_{x1(a)=x1(b)=x1(c)=0} \exp[-i\xi_i k_i'(a)]d\boldsymbol{\xi}=1/(2\pi)^3$$
$$\int C_{ijr}\exp\{i[k_i(a)\xi_i-k_i'(a)\xi_i+k_i(b)\eta_i]\}d\boldsymbol{\xi}\, d\mathbf{k}(a)\, d\mathbf{k}(b)$$
$$= \int C_{ijr}\exp[ik_i(b)\eta_i]\delta[\mathbf{k}(a)-\mathbf{k'}(a)]d\mathbf{k}(a)\, d\mathbf{k}(b)= \int C_{ijr}\exp[ik_i(b)\eta_i]\, d\mathbf{k}(b). \quad (30)$$

This time, operating

$$1/(2\pi)^3\int e\exp[-i\eta_i k_i'(b)]\, d\boldsymbol{\eta}$$

to the both sides of (30), we obtain

$$1/ (2\pi)^6 \int [R_{ijr}^{(1,1,1)}(a, b, c)]_{x1(a)=x1(b)=x1(c)=0} \exp[-i\xi_i k_i'(a)-i\eta_i k_i'(b)]d\boldsymbol{\xi} d\boldsymbol{\eta}$$
$$= \mathbf{1/(2\pi)^3}\int C_{ijr} \exp[ik_i(b)\eta_i-ik_i'(b)]d\boldsymbol{\eta}\, d\mathbf{k}(b)= \int C_{ijr}\exp[ik_i(b)\eta_i-ik_i'(b)]d\boldsymbol{\eta}\, d\mathbf{k}(b)$$
$$=\int C_{ijr}\delta[\mathbf{k}(b)-\mathbf{k'}(b)]d\mathbf{k}(b)=C_{ijr}, \quad (31)$$

That is, if the three-point triple correlations are given experimentally as the function of vectors ξ and **η,** C_{ijr} will be determined. Then, using C_{ijr}, we can obtain E_{ij} and D_{ij} easily, and the wanted particular solution $[R_{ij}^{(1,1)}(a, b)]_p$ in (.25) is derived.

According to the diagram with respect to the turbulent energy spectrum function (Batchelor 1956), most of turbulent energy is concentrated to the narrow range of wave-number, viz. $|\mathbf{k}(a) \ll 1, and\ |\mathbf{k}(b)| \ll 1$. Thus, by expanding D_{ij} and E_{ij} in (25) asymptotically, and examining carefully each of these expansion coefficients, we could find how D_{ij} and E_{ij} depend on the wave-numbers, **k**(a) and, k(b). Having done those tasks as well as substituting D_{ij} and E_{ij}, which are expressed by wave-numbers, in (25), we can find how the particular solution $[R_{ij}^{(1,1)}(a, b)]_p$ varies depending on the main stream-wise coordinate x.

Substituting (23) in the solenoidal condition

$$\partial R_{i,l}^{(1,1)}(a, b)/\partial x_i(a)= \partial R_{i,l}^{(1,1)}(a, b)/\partial x_l(b) =0, \quad (32)$$

we det the following relations,

$$\partial R_{i,l}^{(1,1)}(a, b)/\partial x_i(a)= \partial[R_{i,l}^{(1,1)}(a, b)]_c/\partial x_i(a)+ \partial[R_{i,l}^{(1,1)}(a, b)]_p/\partial x_i(a)=0, \quad (33)$$

And

$$\partial R_{i,l}^{(1,1)}(a, b)/\partial x_l(b)= \partial[R_{i,l}^{(1,1)}(a, b)]_c/\partial x_l(b)+ =0. \quad (34)$$

In general, it is known that any complementary solution multiplied by arbitrary constant is also another complementary solution. In case of any particular solution, the situation is also same. By considering these facts, (33) and (34) provide us the following relations,

$$\partial[R_{i,l}^{(1,1)}(a, b)]_c/\partial x_i(a) = 0, \tag{35}$$

$$\partial[R_{i,l}^{(1,1)}(a, b)]_p/\partial x_i(a)=0, \tag{36}$$

$$\partial[R_{i,l}^{(1,1)}(a, b)]_c/\partial x_l(b)=0, \tag{37}$$

And

$$\partial[R_{i,l}^{(1,1)}(a, b)]_p/\partial x_l(b)=0. \tag{38}$$

For the later convenience, let's rewrite (23) in the form,

$$[R_{ij}^{(1,1)}(a, b)]_p= \int D_{ij} \exp\{-[\beta(a)+\beta(-a-b)-\beta(b)]x+r_1/2\cdot[\beta(a)+\beta(-a-b)-\beta(b)]-ik(b)[\mathbf{x}(a)\mathbf{x}(b)]\}d\mathbf{k}(a)d\mathbf{k}(b)+ \int E_{ij}\exp\{-[\beta(a)+\beta(-a-b)+\beta(b)]x+r_1/2\cdot[\beta(a)-\beta(-a-b)-\beta(b)]+ik(a)[\mathbf{x}(a)-\mathbf{x}(b)]\}d\mathbf{k}(a)d\mathbf{k}(b), \tag{39}$$

Where,

$$x=[x_1(a)+x_1(b)]/2, \text{ and } r_1=x_1(a)-x_1(b).$$

Because relations $x \gg r_1$ and

$$\beta \cong k^2/R \ll 1,$$

we obtain

$$[\beta(a)+\beta(-a-b)-\beta(b)]x \gg r_1/2\cdot[\beta(a)+\beta(-a-b)-\beta(b)],\ r_1/2\cdot[\beta(a)-\beta(-a-b)-\beta(b)]. \tag{40}$$

By using (40), (39) becomes

$$[R_{ij}^{(1,1)}(a, b)]_p= \int D_{ij}\exp\{-[\beta(a)+\beta(-a-b)-\beta(b)]x-ik(b)[\mathbf{x}(a)-x(b)]\}d\mathbf{k}(a)d\mathbf{k}(b)+ \int E_{ij} \exp\{-[\beta(a)+\beta(-a-b) +\beta(b)]x+ik(a)[\mathbf{x}(a)-\mathbf{x}(b)]\}d\mathbf{k}(a)d\mathbf{k}(b). \tag{41}$$

To know how the particular solution depends on the coordinate x, it may be sufficient to obtain the forms of D_{ij} and E_{ij} at x=0. Hence, when x=0, (41) reduces to

$$[R_{ij}^{(1,1)}(a, b)]_p= \int D_{ij} \exp\{-ik(b)[\mathbf{x}(a)-x(b)]\}d\mathbf{k}(a)d\mathbf{k}(b)+ \int E_{ij}\exp\{ik(a)[\mathbf{x}(a)-\mathbf{x}(b)]\}d\mathbf{k}(a)d\mathbf{k}(b). \tag{42}$$

Then, by using (36) and (42), we obtain

$$\partial[Rij(1,1)(a,b)]p/\partial x_i(a)\int[-ik_i(b)]\, D_{ij} \exp\{-ik(b)[\mathbf{x}(a)-x(b)]\}d\mathbf{k}(a)d\mathbf{k}(b)+ \int[ik_i(a)]E_{ij} \exp\{ik(a)[\mathbf{x}(a)- \mathbf{x}(b)]\}d\mathbf{k}(a)d\mathbf{k}(b)=0. \tag{43}$$

As the result, we have the following two conditions regarding to D_{ij} and E_{ij}

$$k_i(b)D_{ij}=0, \quad (44)$$

$$k_i(a)E_{ij}=0. \quad (45)$$

Similarly, to the above, by using (38) and (42), we have

$$k_j(b)D_{ij}=0, \quad (46)$$

$$k_j(a)E_{ij}=0. \quad (47)$$

At this stage, expand D_{ij} near $|\mathbf{k}(a)| = |\mathbf{k}(b)| = \mathbf{0}$ asymptotically, we get

$$D_{ij}[\mathbf{k}(a)\ \mathbf{k}(b)] = D_{ij}^{(0)} + k_k(a)D_{ijk}^{(a)} + k_k(b)D_{ijk}^{(b)} + k_k(a)k_l(a)D_{ijkl}^{(a,a)} + k_k(a)k_l(b)D_{ijkl}^{(a,b)} + k_k(b)k_l(b)D_{ijkl}^{(b,b)} + O(k^3). \quad (48)$$

Then, we have by using (44),

$$k_i(b)D_{ij} = k_i(b)D_{ij}^{(0)} + k_i(b)k_k(a)D_{ijk}^{(a)} + k_i(b)k_k(b)D_{ijk}^{(b)} + k_i(b)k_k(a)k_l(a)D_{ijkl}^{(a,a)} + k_i(b)k_k(a)k_l(b)D_{ijkl}^{(a,b)} + k_i(b)k_k(b)k_l^{(b)}D_{ijkl}^{(b,b)} + O(k^4) = 0. \quad (49)$$

Similarly, to the above, by using (46), we obtain

$$k_j(b)\ D_{ij} = k_j(b)\ D_{ij}^{(0)} + k_j(b)k_k(a)D_{ijk}^{(a)} + k_j(b)k_k(b)D_{ijk}^{(b)} + k_j(b)k_k(a)k_l(a)D_{ijkl}^{(a,a)} + k_{ij}(b)k_k(a)k_l(b)D_{ijkl}^{(a,b)} + k_j(b)k_k(b)k_l(b)D_{ijkl}^{(b,b)} + O(k^4) = 0. \quad (50)$$

(49) and (50) give us the following relations regarding to expansion coefficients of D_{ij}.

$$D_{ij}^{(0)}=0, \quad (51)$$

$$D_{ijk}^{(a)}=0, \quad (52)$$

$$D_{ijk}^{(b)}=\varepsilon_{ijk}D^{(b)}+\varepsilon_{jik}D^{(b)*}, \quad (53)$$

where $D^{(b)*}$ is the complex conjugate of $D^{(b)}$. ε_{ijk} is the alternating tensor, where ε_{ijk}=0, 1, or -1 when suffixes are not all different, in cyclic order or not in cyclic order, respectively. In addition, we use the following relations,

$$D_{ijk}^{(b)}+D_{kji}^{(b)}=0, \quad (54)$$

$$D_{ijk}^{(b)}+D_{ikj}^{(b)}=0, \quad (55)$$

where the tensor $D_{ijk}^{(b)}$ is Hermitian with respect to subscripts I and j,

$$D_{ijk}^{(b)}=D_{jik}^{(b)*}, \quad (56)$$

$$D_{ijkl}^{(a,a)}=\varepsilon_{klm}\hat{D}_{ijm}, \quad (57)$$

where we use the condition

$$D_{ijkl}^{(a,a)}+ D_{ijlk}^{(a,a)}. \quad (58)$$

And

$$D_{ijkl}^{(a,b)}=\varepsilon_{ijl}\hat{D}_k^{(a,b)}+\varepsilon_{jil}\hat{D}_k^{(a,b)*}, \quad (59)$$

where we use the following relations,

$$D_{ijkl}^{(a,b)}+D_{ljki}^{(a,b)}=0, \quad (60)$$

And

$$D_{ijkl}^{(a,b)}+D_{ilkj}^{(a,b)}=0, \quad (61)$$

where tensor $D_{ijkl}^{(a,b)}$ is Hermitian with respect to subscripts I and j, so that

$$D_{ijkl}^{(a,b)}=D_{jikl}^{(ab)*}. \quad (62)$$

Now, consider how tensor $D_{ijkl}^{(b,b)}$ can be described. (49) and (50) give us immediately the following two relations

$$k_i(b)k_k(b)k_l(b)D_{ijkl}^{(b,b)}=0, \quad (63)$$

and

$$k_j(b)k_k(b)k_l(b)D_{ijkl}^{(b,b}=0. \quad (64)$$

Let’s introduce the second-order tensor as follow,

$$A_{ij}= k_k(b)k_l(b)D_{ijkl}^{(b,b)}, \quad (65)$$

Since (63) and (65) give us $k_i(b)A_{ij}=0$, it may be possible to express A_{ij}, as

$$A_{ij}=\varepsilon_{ipq}k_p(b)C_{qj}. \quad (66)$$

Because A_{ij} is a linear quadratic form of the components in **k**(b), C_{qj} may be expressed by

$$C_{qj}=\Gamma_{qjr}k_r(b), \quad (67)$$

Thus, substituting (67) in (66), we have

$$A_{ij}=\varepsilon_{ipq}\Gamma_{qjr}k_p(b)k_r(b). \quad (68)$$

Similarly, to the above,

$$A_{ij}=\varepsilon_{ipq}\Gamma'_{qip}k_p(b)k_r(b). \quad (69)$$

Because (65) and (68) must be valid for all **k**(b), the former relation can be rewritten as

$$A_{ij}=k_p(b)k_r(b)D_{ijpr}^{(b,b)}. \quad (70)$$

Hence, (67)-(69) give us

$$D_{ijpr}^{(b,b)}= \varepsilon_{ipq}\Gamma_{qjr}=\varepsilon_{jrq}\Gamma'_{qip}. \quad (71)$$

Note that the last term in (71) becomes zero when r=j, for it is antisymmetric with respect to the interchange in r and j. It is, therefore, required that Γ_{qjr} has those properties, so it can be expressed by

$$\Gamma_{qjr} =\varepsilon_{jrb}\hat{D}_{qb}^{(b,b)}, \quad (72)$$

where $\hat{D}_{qb}^{(b,b)}$ is an arbitrary tensor. (71) and (72) give us the general form of $D_{ijkl}^{(b,b)}$ as follow,

$$D_{ijkl}^{(b,b)}=\varepsilon_{ika}\varepsilon_{jlb}\hat{D}_{ab}^{(b,b)} . \quad (73)$$

Subtituting (51)-(61) and (73) in (48), we get

$$D_{ij}=k_k(b)(\varepsilon_{ijk}D^{(b)}+\varepsilon_{jik}D^{(b)*})+k_k(a)k_l(a)\varepsilon_{klm}\hat{D}_{ijm}^{(a,a)}+ k_k(a)k_l(b)(\varepsilon_{ijl}\hat{D}_k^{(a,b)}+\varepsilon_{jil}\hat{D}_k^{(a,b)*})+ k_k(b)k_l(b)\epsilon_{ika}\epsilon_{jlb}\hat{D}_{ab}^{(b,b)}+O(k^3). \quad (74)$$

At this stage of the analyses, it is necessary for us to quote Cramér's theorem (1940), to specify the coefficients of $k_k(b)$ in (74) to become zero. His theorem may be stated in such a way "the necessary and sufficient condition that $R_{ij}(\mathbf{r})$ is the correlation tensor of a continuous stationary random process is that it can be expressed of the form

$$R_{ij}(\mathbf{r})=\int \Phi_{ij}(\mathbf{k}) \exp(ik_ir_i)d\mathbf{k},$$

where $\Phi_{ij}(\mathbf{k})$ is a complex tensor that satisfies the following two conditions,

$$\text{(a)} \qquad \int |\Phi_{ij}(\mathbf{k})|\, d\mathbf{k}< \infty,$$

and

$$\text{(b)} \qquad \Phi=x_ix_j^*\Phi_{ij}(\mathbf{k}),$$

is a non-negative quadratic form. That is, $\Phi \geq 0$ for an arbtrary choice of the complex constants x_i. "In his theorem, dk is written by $dk_1dk_2dk_3$, the integrals are taken over the whole wave-number space, and x_i^* denotes the complex conjugate of x_i. The behavior of the spectrum $\Phi_{ij}(k)$ at small values of k may be determined with the aid of his theorem and the incompressible condition. We may be required to assume that a few of the first derivatives for $\Phi_{ij}(k)$ at k=0 exist. It is known that stationary random functions do not necessarily satisfy such conditions in general, but some experiments confirm the validity of the present assumption in case of homogeneous turbulence.

In the neighborhood of k=0, the spectrum tensor can be expressed by

$$\Phi_{ij}(\mathbf{k}) = B_{ij} + k_k B_{ijk} + k_k k_l B_{ijkl} + O(k^3), \tag{75}$$

where the tensor coefficients Bij ,B_{ijk} , and B_{ijkl} depend on time only. The incompressible condition, $k_i\Phi_{ij}(\mathbf{k})=k_j\Phi_{ij}(\mathbf{k})=0$ requires

$$k_i B_{ij} + k_i k_k B_{ijk} + k_i k_k k_l B_{ijkl} + O(k^4) = 0, \tag{76}$$

being satisfied by every vaue of k only if B_{ij} =0. Then, noting that Cramér's theorem assures us

$$x_i x_j^* k_k B_{ijk} \geq 0 \tag{77}$$

for all sufficiently small k and arbitrary x_i. Because the sign of (77) could be altered by reversing the direction of k, the only possibility is B_{ijk}=0. The expression of $\Phi_{ij}(k)$ in the vicinity of k=0 must be

$$\Phi_{ij}(\mathbf{k}) = k_k k_l B_{ijkl} + O(k^3). \tag{78}$$

The energy spectrum tensor of $R_{ij}^{(1,1)}$ (a,b) in (23) is no more than $\Phi_{ij}(k)$. The tensor coefficient of the first-order wavenumber corresponding to B_{ijk} in the above relation, becomes zero. On one hand, the energy spectrum tensor C_{ij} of the complementary solution has also not the tensor coefficient for the first-order wave-number, as already being showed in (4-20) in part 1 of the present paper. It is, therefore, evident that the tensor coefficient of $k_k(b)$ in (74) reduces to zero. That is,

$$D_{ij}=k_k(a)k_l(a)\varepsilon_{klm}\hat{D}_{ijm}^{(a,a)}+ k_k(a)k_l(b)(\varepsilon_{ijl}\hat{D}_k^{(a,b)}+\varepsilon_{jil}\hat{D}_k^{(a,b)*})+k_k(b)k_l(b)\epsilon_{ika}\epsilon_{jlb}\hat{D}^{(b,b)}+O(k^3). \tag{79}$$

In particular, when i=j for the energy spectrum of turbulent intensity, we obtain

$$D_{ii} = k_k(a)k_l(a)\varepsilon_{klm}\hat{D}_{iim}^{(a,a)}+k_k(b)k_l(b)\epsilon_{ika}\epsilon_{ilb}\hat{D}_{ab}^{(b,b)}+O(k^3). \tag{80}$$

Similarly to D_{ij} , expanding E_{ij} in the vicinity of $|\mathbf{k}(a)| = |\mathbf{k}(b)| = 0$ in terms of conditions (45) and (47), we get the following relations with respect to the expansion coefficients,

$$E_{ij}^{(0)}=0, \tag{81}$$

$$E_{ijk}^{(b)}=0, \tag{82}$$

$$E_{ijk}^{(a)}=\varepsilon_{ijk}E^{(a)} + \varepsilon_{jik}E^{(a)*}, \tag{83}$$

$$E_{ijkl}^{(b,b)} = \varepsilon_{klm}\hat{E}_{ijm}^{(b,b)}, \tag{84}$$

$$E_{ijkl}^{(a,b)} = \varepsilon_{ijk}\hat{E}_l^{(a,b)} + \varepsilon_{jik}\hat{E}_l^{(a,b)*}, \tag{85}$$

$$E_{ijkl}^{(a,a)} = \varepsilon_{ika}\varepsilon_{jlb}\hat{E}_{ab}^{(a,a)}. \tag{86}$$

Based on Cramér's theorem, (83) becomes zero, viz.

$$E_{ijk}{}^{(a)}=\varepsilon_{ijk}E^{(a)} + \varepsilon_{jik}E^{(a)*} =0. \quad (87)$$

Hence, the concrete expression of E_{ij} may be expressed by

$$E_{ij} = k_k(b)k_l(b)\varepsilon_{klm}\hat{E}_{ijm}{}^{(b,b)} + k_k(a)k_l(b)(\varepsilon_{ijk}\hat{E}_l{}^{(a,b)}+\varepsilon_{jik}\hat{E}_l{}^{(a,b)*})+ k_k(a)k_l(a)\varepsilon_{ika}\varepsilon_{jlb}\hat{E}_{ab}{}^{(a,a)} + O(k^3). \quad (88)$$

In particular, when i=j, corresponding the power spectrum of the turbulent intensity,

$$E_{ii} = k_k(b)k_l(b)\varepsilon_{klm}\hat{E}_{iim}{}^{(b,b)} + k_k(a)k_l(a)\varepsilon_{ika}\varepsilon_{ilb}\hat{E}_{ab}{}^{(a,a)} + O(k^3). \quad (89)$$

To derive the turbulent intensity relating to the particular solution, by definition, put i=j,

$$\dot{u}_p{}^2 =[R_{ii}{}^{(1,1)} (a, b)]_p =\int D_{ii}\exp\{-2x/R[k^2(a)+k_i(a)k_i(b)+k^2(b)]\}d\mathbf{k}(a)d\mathbf{k}(b)+ \int E_{ii} \exp\{-2x/R[k^2(a)+k_i(a)k_i(b)+k^2(b)]\}d\mathbf{k}(a)d\mathbf{k}(b), \quad (90)$$

where $\dot{u}_p{}^2$ is the turbulent intensity due to the particular solution. Now, substituting (80) and (89) in (90), we get

$$\dot{u}_p{}^2 =\int [(\varepsilon_{klm}\hat{D}_{iim}{}^{(a,a)}+ \varepsilon_{ika}\varepsilon_{ilb}\hat{E}_{ab}{}^{(a,a)})k_k(a)k_l(a) +(\epsilon_{ika}\epsilon_{ilb}\hat{D}_{ab}{}^{(b,b)}+ \varepsilon_{klm}\hat{E}_{iim}{}^{(b,b)})k_k(b)k_l(b)]\cdot \exp\{-2x/R[k^2(a)+k_i(a)k_i(b)+k^2(b)]\}d\mathbf{k}(a)d\mathbf{k}(b), \quad (91)$$

where we use the relation $O(k^2) \gg O(k^3)$. For the sake of future convenience, rewrite (91) as

$$\dot{u}_p{}^2 =\int [G^{(a,a)}k_k(a)k_l(a) +H^{(b,b)}k_k(b)k_l(b)]\cdot \exp\{-2x/R[k^2(a)+k_i(a)k_i(b)+k^2(b)]\}d\mathbf{k}(a)d\mathbf{k}(b), \quad (92)$$

with

$$G^{(a,a)}= \varepsilon_{klm}\hat{D}_{iim}{}^{(a,a)}+ \varepsilon_{ika}\varepsilon_{ilb}\hat{E}_{ab}{}^{(a,a)}, \quad (93)$$

And

$$H^{(b,b)}= \epsilon_{ika}\epsilon_{ilb}\hat{D}_{ab}{}^{(b,b)}+ \varepsilon_{klm}\hat{E}_{iim}{}^{(b,} . \quad (94)$$

Let's rewrite (92) as follow,

$$\dot{u}_p{}^2 =\int [G^{(a,a)}k_k(a)k_l(a) +H^{(b,b)}k_k(b)k_l(b)]\cdot \exp\{-2x/R\cdot[k_i(a)+k_i(b)/2]^2-3xk^2(b)/(2R)\}d\mathbf{k}(a)d\mathbf{k}(b), \quad (95)$$

Finally, let's us change variables in (95) in the following manner,

$$V_i=k_i(b), \quad (96)$$

And

$$W_i=k_i(a) + k_i(b)/2=k_i(a)+V_i/2. \quad (97)$$

Thus, the Jacobian becomes as

$$J_1=\partial(W_1, W_2, W_3, V_1, V_2, V_3)/\partial[k_1(a), k_2(a), k_3(a), k_1(b), k_2(b), k_3(b)] =1. \quad (98)$$

Using (96)-(98), we can rewrite (95) of the form,

$$\bar{u}_p{}^2 =\int [G^{(a,a)}(W_kW_l-W_kV_l/2-V_kW_l/2+V_kV_l/4) +H^{(b,b)}V_kV_l]\cdot \exp\{-2x/R\cdot W^2-3xV^2/(2R)\}d\mathbf{W}d\mathbf{V}. \quad (99)$$

Having done this preparation, let's change the variables in (99) from Wi and V_i to w_i and v_i, respectively, as follows,

$$W_i =1/2\cdot (R/x)^{1/2} w_i, \quad (100)$$

and

$$V_i=[R/(3x)]^{1/2} v_i. \quad (101)$$

The Jacobian can be expressed by

$$J_2 =\partial(W_i, V_i)/\partial(w_i, v_i) =$$

$$\begin{vmatrix} \partial W_i/\partial w_i & \partial W_i/\partial v_i \\ \partial V_i/\partial w_i & \partial V_i/\partial v \end{vmatrix}$$

With (100) and (101), we can calculate each of the elements in the above determinant as follow.

$$\begin{vmatrix} 1/2\cdot (R/x)^{1/2} & 0 \\ 0 & [R/(3x)]^{1/2} \end{vmatrix}$$

$$= 1/2\cdot (R/x)^{1/2} \cdot [R/(3x)]^{1/2}$$

$$=1/2\cdot 1/3^{1/2}\cdot(R/x)$$

$$=(1/12)^{1/2}\cdot(R/x)$$

Thus, finally Jacobian becomes

$$J_2 = (1/12)^{1/2}\cdot(R/x). \quad (102)$$

By using (100) – (102), we can rewrite (99) as follow,

$$\bar{u}_p{}^2 =\int [G^{(a,a)}([1/2\cdot (R/x)^{1/2}]^2 w_kw_l-1/4\cdot (R/x)^{1/2}[R/(3x)]^{1/2}w_kv_l-1/4\cdot (R/x)^{1/2}[R/(3x)]^{1/2}v_kw_l+ R/(3x)/4) v_kv_l+ H^{(b,b)}(R/(3x) v_kv_l]\cdot \exp(w2/2-v2/2)[1/2\cdot(R/x)1/2[R/(3x)]dw.\ dv, \quad (103)$$

Now, let's define Hermite polynomials in the following way,

$$w_i = ₤_i^{(1)}(w), \quad (104)$$

$$v_i = ₤_i^{(1)}(v), \quad (105)$$

$$w_i w_j - \delta_{ij} = ₤_{ij}^{(2)}(w), \quad (106)$$

$$v_i v_j - \delta_{ij} = ₤_{ij}^{(2)}(v). \quad (107)$$

Then, substituting (104)-(107) in (103), we get

$$\bar{u}_p^2 = \int 【G^{(a,a)}\{ [1/2\cdot (R/x)^{1/2}]^2 [₤_{kl}^{(2)}(w)+\delta_{kl}]-1/4\cdot (R/x)^{1/2}[R/(3x)]^{1/2}₤_k^{(1)}(w)\, ₤_l^{(1)}(v)- 1/4\cdot (R/x)^{1/2}[R/(3x)]^{1/2}[₤_k^{(1)}(v)₤_l^{(1)}(w)]+R/(3x)/4)\, [₤_{kl}^{(2)}(w)+\delta_{kl}]】 +H^{(b,b)}[R/(3x)\, [₤_{kl}^{(2)}(w)+\delta_{kl}]\cdot \exp(w^2/2-v^2/2)[1/2\cdot(R/x)^{1/2}[R/(3x)]^{1/2}d\mathbf{w}.d\mathbf{v}. \quad (108)$$

Moreover, using the orthogonality of Hermite function, together with the following relations

$$\int ₤_{kl}^{(2)}(w)\cdot\exp(-w^2/2)\, d\mathbf{w} = \int ₤_{kl}^{(2)}(v)\cdot\exp(-v^2/2)\, d\mathbf{v} = \mathbf{0}, \quad (109)$$

(108) becomes

$$\bar{u}_p^2 = 2\pi^3/(93^{1/2})\cdot \delta_{kl}(G^{(a,a)} + H^{(b,b)})(R/x)^4, \quad (110)$$

Recalling the notation $G^{(a,a)}$ and $H^{(b,b)}$, we obtain the final form;

$$\bar{u}_p^2 = 2\pi^3/ (93^{1/2})\cdot \delta_{kl}((\hat{E}_{aa}{}^{(a,a)} + \hat{D}_{aa}{}^{(b,b)})R^4x^{-4}, \quad (111)$$

Finally, referring to (23) for the decay law $\bar{u}_c^2$ for turbulent energy in the final period as well as (111), we have the decay law for turbulent energy in the transitional period as follow

$$\bar{u}^2 = \bar{u}_c^2 + \bar{u}_p^2 = 3a^2/32\cdot(R/2)^{5/2}\cdot x^{-5/2} + 2\pi^3/(93^{1/2})\cdot \delta_{kl}((\hat{E}_{aa}{}^{(a,a)} + \hat{D}_{aa}{}^{(b,b)})R^4x^{-4}, \quad (112)$$

Comparison between the present theory with an experiment

Figure 2 shows the comparison between the present theory with the experiment by Batchelor-Townsend (1948). The curve 1 is the present theory of the decay law for turbulent energy in the final period, which is expressed by (4.23) in Part 1. This problem has been solved exactly as an initial value problem based on reliable data as depicted in Figure 3.

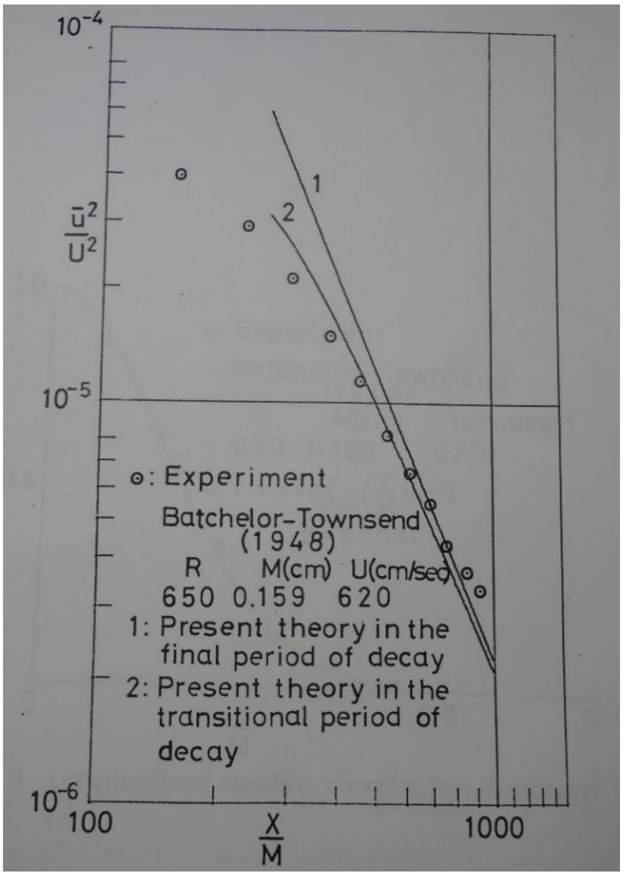

Figure 2: Energy decay of grid-produced turbulence.

The comparison is done for the final period corresponding to x/M≥ 600 in the experiment. It is evident that the curve 1 is in good agreement with the experimental data for this period. In terms of Taylor's hypothesis, the present decay law for turbulent energy $\bar{u}^2$ depending on $x^{-5/2}$ in the final period is consistent with the classical one. Present curve 2 that is expressed by (112) represents the decay law for turbulent energy in the transitional period, for it includes effects of the double as well as triple correlations,

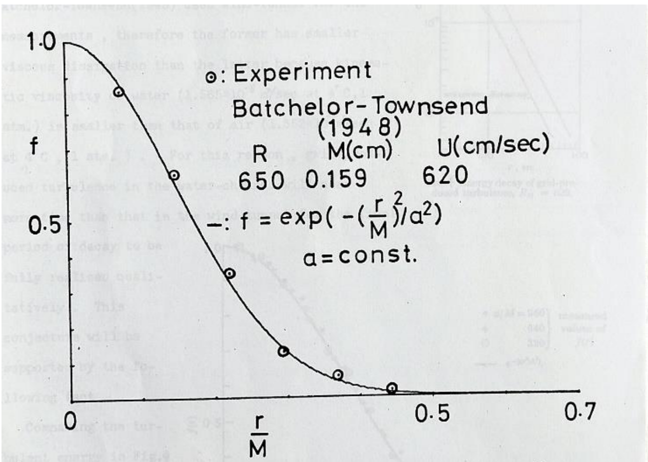

Figure 3: Longitudinal double correlation function.

a = 0.208934

but neglects the quadruple correlations or assumes four-particle molecular chaos. This curve agrees well with the experimental data corresponding to x/M≥ 460. However, it must be noted that owing to the deficit of the experimental data for the triple correlations at the initial plane behind the grid, we could not determine the constant value in front of x^4 in (112). In another words, the mathematical expression of the triple correlations to be used as the initial condition required in the present theory does not exist currently as far as the present author knows. Thus, such an experiment is strongly required to get the expression of triple correlations. So, this work is left for the future, but for tentative comparison with the data on the turbulent intensity $\overline{u}^2$, the value of B is set to be of $-1.5x10^5$. In another words, this is the assumption that both of the double correlation and the triple correlation are given at the same plane behind the grid.

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Appendix 1: Determinant

If in the determinant |a|, we delete the ith row and jth column, and form a determinant from all the elements remaining, we shall have a new determinant of n-1 rows and columns. This new determinant is defined to be the minor of the element a_{ij}. For example, if |a| is a determinant of the third order, the minor of the element a_{32} is denoted by M_{32}.

The cofactor of an element of a determinant a_{ij} is the minor of that element with a sign attached to it determined by the numbers i and j which fix the position of a_{ij} in the determinant |a| The sign is chosen by the equation

$$A_{ij} = (-1)^{i+j} M_{ij},$$

where A_{ij} is the cofactor of the element a_{ij} and M_{ij} is the minor of the element a_{ij}.

In case of the nth-order determinant, as the unique nth order homogeneous polynomial

|a| is given by

$$|a| = \sum_{j=1}^{n} a_{ij} A_{ij},$$

where the a_{ij} quantities must be taken either from a single row or a single column. In this case the cofactors A_{ij} are determinants of the (n-1) st order, but they may be in turn expanded by the above rule, and so forth, until the result is a homogeneous polynomial of the nth order.

Appendix 2: Hermite Function

The function $\varphi_n(x)=\exp(-x^2/2)\,Hn(x)$ (n=0, 1,2, ...), which are often referred to as Hermite functions, satisfy the differential equation

$$d^2w/dz^2 + (2n+1-z^2)\,w = 0, \quad (n=0, 1, 2, \ldots)$$

and

$$\int_{-\infty}^{\infty} \varphi_n(x)\varphi_{n'}(x)dx = 2^n n!\, \pi^{1/2} \delta_{n'}{}^{n}.$$