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On the Inner Automorphisms and Central Automorphisms of Nilpotent Group of Class 2 Which Fix the Centre Elementwise

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Abstract:
Suppose that G is a finite p-group. It was shown (see [2]) that C∗ the set of all central automorphisms of G which elementarily fixes the centre of G elementwise, is isomorphic to the group of all Inner automorphisms of G if and only if G is abelian or G is nilpotent of class 2 for which the centre of G is cyclic. More so, if G is finitely generated then G can be represented in a particular simple form (see [3]). Moreover, suppose that G is a finite p-group such that Aut(G) ≡ $E_p{}^m$.Then, C_{Autc} (G)(Z(G)) = G/Z(G).

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INTRODUCTION

We assume that p denotes a prime number. Suppose that G_1 and G_2 are two groups. Denote the set of allgroup homomorphisms from G_1 to G_2 by *Hom* (G_1, G_2). It is noteworthy that if G_2 is abelian, then it induces commutativity on *Hom* (G_1, G_2) and we define the binary operation by $(f_1 f_2)x = f_1(x) f_2(x) \forall f_1, f_2 \epsilon Hom(G_1, G_2)$ and all x ϵ G. By definition: An automorphism ϕ of G is known as a central automorphism if $x^{-1}\phi(x)$ ϵ Z(G) for each x ϵ G. A normal subgroup $Aut_c(G)$ of the full automorphism group of G is formed by the central automorphisms of G. Various other contributions have been made through a quite appreciable number of algebraists (see [1], [4]) .The finite p-groups G for which the central automorphisms of G equals the Inner automorphisms were discussed and generally characterised at length by Curran M.J. and McCaughan D.J. (see [4]) .It was shown that if G is a finite p-group , then $Aut_c(G)$ = Inn(G) iff G' = Z(G)and Z(G) is cyclic . Mehdi Shabani Attar (see [2]) has proved that: if G is a finite p-group, then $C_{Aut_C(G)}(Z(G))$ = Inn(G) iff G is abelian or G is nilpotent of class 2 and Z(G) is cyclic.

Proof

First of all, it may be shown that $C_{Aut_C(G)}(Z(G)) \doteq Hom(G/Z(G), Z(G))$ Consider that every element of $C_{Aut_C(G)}(Z(G))$ fixes each element of Z(G), for each g ϵ $C_{Aut_C(G)}(Z(G))$ the map αg: $G/Z(G) \rightarrow Z(G)$ defined by $\alpha g(bZ(G)) = b^{-1}g(b)$. This is a well-defined mapping and hence a homomorphism Consider that α: $g \rightarrow \alpha g$ is a monomorphism. Thus, for each t ϵ $Hom(G/Z(G)$, $Z(G))$, the map g defined by g(b) = bt(bZ(G)) ∀ b ϵ G, is a central automorphism which fixes Z(G) elementwise, and $\alpha(g)$ = k. Hence α is a group isomorphism and $C_{Aut_C(G)}(Z(G))$ is isomorphic to Hom(G/Z(G), Z(G)) Now, if $C_{Aut_{C(G)}}(Z(G))$ = Inn(G) and G is nonabelian. Suppose that b G ϵ

Then $\theta_b \in$ Inn(G) induced by b is a central automorphism. Thus, $[x, b] = x^{-1}\theta_b(x) \in$ Z(G) ∀ x ∈ G. By this, G is nilpotent and of class 2.

Thus, exp(G/Z(G)) = exp(G') = p^k, k ∈ N. Now, let G/Z(G) and Z(G) have rank u and v respectfully. Since G is of class 2 and nilpotent, we have that $| G/Z(G) | = | Inn(G) | = | C_{Aut_{C(G)}}(Z(G)) | = | Hom(G/Z(G), Z(G)) \geq | G/Z(G) | p^{u(v-1)}$. Now, u ≥ 2, v = 1, then Z(G) is cyclic. On the other hand, if G is abelian, then $C_{Aut_{C}(G)}(Z(G))$ = Inn(G) = 1. Now, since G/Z(G) is abelian and of exponent G', where G' is cyclic thenby [2], Hom(G/Z(G), Z(G)) is isomorphic to G/Z(G).

Thus, $C_{Aut_{C}(G)}(Z(G)) \approx$ Hom(G/Z(G), Z(G)). But then, G/Z(G) is isomorphic to Inn(G). Thus, for thefact that G is of class 2 and a nilpotent group, Inn(G) ≤ $C_{Aut_{C}(G)}(Z(G))$ = Inn(G)

Theorem
(See [3]) Let G be a finitely generated nilpotent group of class 2. Then $C^* \simeq$ Inn(G) iff Z(G) is cyclic or Z(G) $\simeq C_m \times Z^r$ where G/Z(G) has exponent dividing m and r is torsion-free rank of Z(G).

Lemma A
Let X, Y, and Z be abelian groups. Then,

1. Hom(XxY , Z) ≃ Hom(X, Z)xHom(Y,Z)
2. Hom(X, YxZ) ≃ Hom(X,Y)xHom(X,Z) .
3. $Hom(C_m, C_n) \simeq C_d$,where d = gcd(m , n) .
4. Hom(Z , X) ≃ X .

Proposition
Suppose that X and Y are 2 finite abelian p-groups and exp(Y) is divisible by exp(X). Then Hom(X , Y) ≃ X iff Y is cyclic .

Proof
If Y is a cyclic group, then exp(Y) =$|Y|$. By lemma A (i) & (iii), Hom(X , Y) ≃ X .Conversely , let Hom(X , Y) ≃ X . In order to obtain a contradiction, suppose that Y is notcyclic, then we have Y≃ $C_{p^i} x Z$, where exp(Y) = p^i and Z is a nontrivial abelian group. By lemma A (i) & (iii), Hom(X , Y) ≃ XxHom(X , Z) . By assumption, we have X≃ XxHom(X , Z) . Thus, Hom(X , Z) = 1 ⇒⇐, ∴ Y is cyclic.

Corollary
Let X and Y be 2 finite abelian groups and exp(Y) is divisible by exp(X). Then, Hom (X, Y) ≃ X iff Y is a cyclic group.

Proposition
Let G be a group. Then,

(a) If Z(G) ≤ G', then $Aut_c(G)$ $Hom(G/G', Z(G))$ Hom(G , Z(G))
(b) $C^* \simeq Hom(G/G'Z(G), Z(G)) \simeq Hom(G/Z(G), Z(G))$.

Theorem
Let G be a finitely generated nilpotent group of class 2. Then, Hom(G/Z(G), Z(G)) ≃ G/Z(G) iffZ(G) is cyclic or Z(G) $\simeq C_m \times Z^r$ where G/Z(G) has exponent dividing m and r is torsion-free rank of Z(G).

Lemma

Let G be a finitely generated nilpotent group of class 2 and G/Z(G) = $< g_1Z(G) >$ x $\cdots$ x $< g_tZ(G) >$, for some $g_1, \cdots g_t \in$ G. Then,

(a) $G' = < [g_i, g_j]: 1 \leq i < j \leq t >$.
(b) If G/Z(G) is torsion-free, then G' is torsion and exp(G/Z(G)) = exp(G').
(c) T(G/Z(G)) has exponent dividing exponent of T(Z(G)), where T(G) is the torsion subgroup of G.

Suppose that G is not torsion-free, then the following corollary expresses a satisfactory classification.

Corollary B

Let G be a finitely generated group which is not torsion-free. Then C^* = Inn(G) iff G is nilpotent of class 2. and Z(G) is cyclic or Z(G) $\simeq C_v$ x Z^t where G/Z(G) has exponent dividing v and t is the torsion-free rank of Z(G)

Corollary

Let G be a finitely generated nilpotent group of class 2. Then, G' is torsion-free and $C^* \simeq$ Inn(G)iff Z(G) is infinitely cyclic.

Attar and McCaughan (see [2] and [4]) presented some very useful results as follows:

Corollary

Suppose that G is a finite *p*-group. Then,

(a) C^* = Inn(G) iff G is abelian or G is nilpotent of class 2 and Z(G) is cyclic.
(b) $Aut_c(G)$ = Inn(G) iff G' = Z(G) and Z(G) is cyclic.

Proof

(a) If C^* = Inn(G) and G is non-abelian, then G is finite nilpotent group of class 2 and by corollary (B) Z(G) is finitely cyclic. On the other hand, if G is nilpotent of class 2 and Z(G) is cyclic, then the result follows by corollary (B)
(b) Assume that $Aut_c(G)$ = Inn(G) ∴ C^* = Inn(G) and so, by (a), Z(G) is cyclic. And since $Aut_c(G)$ = C^*, then $|Hom(G/G', Z(G))|$=$|Hom(G/Z(G), Z(G))|$. Hence, $G/G' \simeq$ G/Z(G), satisfying the assertion that: G' = Z(G). If G' = Z(G) and Z(G) is cyclic, then $Aut_c(G) = C^*$. But C^* = Inn(G), since Z(G) is cyclic. ■

MAIN RESULT

Suppose that *G* is a finite *p*-group such that $Aut(G) \equiv E_p^m$.Then, $C_{AutC}(G)(Z(G)) = G/Z(G)$.

STATEMENT OF PROOF OF MAIN RESULT

Definition

Metacyclic Group: This is a group *G* such that both the derived subgroup G' and the quotient group G/G' are cyclic. Such group has a cyclic normal subgroup *L* such that *G/L* is also cyclic.

By the question posed by Y. Berkovich, (See [6]) on whether it is possible to put some general structure on the group *G* in which $Aut(G) = E_p{}^m$, we have that in such a case, $Inn(G) \equiv G/Z(G)$ is also elementary abelian.

Theorem

Jafary(See [6]) Let G be a finite purely non-abelian p-group where p is odd . Then, $Aut_c(G)$ is an elementary abelian p-group iff $exp(Z(G)) = p$ or $exp(G/G') = p$. By this we have further, the following theorem.

Theorem

Suppose that G is a finite p-group such that $Aut(G) = E_p{}^m$.

Then, one of the following holds.

(a) $Z(G) = \Phi(G) = E_p{}^m$.

(b) $G' = \Phi(G)$.

Applying these theorems, we have that if G is abelian or G is nilpotent of class 2 and Z(G) is cyclic (whence $Z(G) \equiv G'$) then, as we have before by Mehdi Shabani Attar that this condition implies that $C_{AutC}(G)(Z(G)) = Inn(G)$, we have that:

1. $C_{AutC}(G)(Z(G)) = G/Z(G)$ and
2. $C_{AutC}(G)(Z(G))$ is metacyclic if Z(G) is cyclic.

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