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A Tutorial Excerpt on the Graeffe's Roots-Finding Numerical Scheme Derivation and Application for the Solution of Third-Degree Polynomial of the form, f(x) = 0

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Abstract:
The research is concerned with the presentation of a tutorial extract on the Graeffe's Root-finding scheme derivation and application for the solution of third-degree Polynomial of the form, f(x) = 0. Thereafter, we tested the efficiency of our proposed scheme by applying it to a range of Third-Degree polynomial problems in literature reviewed. The outcome of the comparison of the roots generated by the Graeffe's root-finding scheme to their respective exact solution showed that the scheme gave a better approximation to the exact solution at every fourth iteration. Thus, the proposed scheme in this research can be said to be another better suitable numerical approach for the solution of third-degree polynomial that their exact solutions are difficult to arrive at. The procedures for the scheme derivation can be easily followed for the solution of other higher degree auxiliary equations.

Keywords: third degree Polynomial, Tutorial Excerpt, Graeffe's Numerical Scheme, Comparison with Exact Solution, Comparison with Newton Raphson's Method, Graphical Profiles, Scheme Derivation, Iterations.

INTRODUCTION

Most differential equations' auxiliary polynomials and some other polynomials are indeed tricky to address in the world of science and engineering today (especially those above second degrees and those that cannot be easily factorised). Thus, various methods that can suit the need of scientists have been evolving. Historically, over some decades ago till date, the exact solution to such polynomials of degree three or anyone higher, has been solely tied to trial-and-error (or guess) method. But this trial-and-error approach however, appears almost not scientifically inclined. Hence, in order to salvage this situation, a numerical method proposed by Graeffe Dandlin becomes unavoidable since it provides all the n-number of roots to any polynomial of degree n. According to [2], the Graeffe's Numerical Method of finding the roots of polynomials is one among many methods recommended for the numerical solution of polynomial equations. This Graeffe root finding method gives all the roots approximations from the first to the last iteration [8]. It is one of the direct roots-finding-methods in literature.

One of the motivations for this research emanates from the fact that in the field of engineering and computational sciences, most differential equations cannot be successfully solved without firstly resolving their reduced auxiliary equations via the help of any root-finding algorithms. The situation becomes even worst when great scientists are handicapped when auxiliary equations cannot be easily factorised via trial-and-error approach. Hence the need for escape route just as this study has proposed cannot be avoided.

Similarly, this method does not require any initial guesses for roots. Going down the memory lane, the Graeffe root finding method was invented independently by Graeffe Dandlin and Lobachevesky in the 19th and 20th century [6,3]. It was seen to be the most popular numerical method for finding roots of polynomials; [7,4,5]. In view of the fact that most numerical methods have limitations and shortcomings, the limitations of the Graeffe's root finding algorithm are manageable and even avoided in an efficient implementation, according to [6]. Additionally, the beauty of the Graeffe's root finding algorithm was likewise demonstrated by [11] for only second-degree polynomials and in [13]. But the third-degree polynomials have not been widely explored in the directions recommended by this study.

However, despite the good sides of the Graeffe's algorithm, according to [1], special cases in Graeffe's method exists such that, if maximum power of polynomial is odd and after squaring, if any coefficient of the function except the constant term, is zero, the method does not give exact roots. An example of such problem that belongs to this category is: if $f(x) = x^3-1= 0$.

DERIVATION OF THE GRAEFFE'S ROOTS SQUARING NUMERICAL SCHEME FOR THE SOLUTION OF THIRD ORDER POLYNOMIAL OF THE FORM, $f(x)=0$

In order to derive the numerical scheme for this study, some numbers of iterations are needed. The higher the number of iterations made for a certain problem, the better the numerical result of the roots of the polynomials in comparison with the exact solution. Generally, according to [1], a minimum of four iterations is almost sufficient to arriving at a better approximation to the exact solution. Therefore, in this numerical scheme derivation subsection, we intend to stop after the fourth iteration. The following are the iterations for the complete scheme/algorithm presented in equations (17), (18) and (19).

First Iteration

Recalling from the general problem of the form, (1)

$$f(x)=0 \tag{1}$$

Now, from equation (1), if the polynomial is of degree 2, the roots will definitely correspond to the roots obtainable using quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}: \tag{2}$$

Since our interest is for third order polynomials, we considered the polynomial whose highest power/degree is odd as shown in equation (3) below.

$$f(x)=ax^{2n+1}+bx^{2n}+cx^{2n-1}+dx^{2n-2} \tag{3}$$

Likewise, setting f(x)=0 in equation (3) above gives;

$$ax^{2n+1}+bx^{2n}+cx^{2n-1}+dx^{2n-2}=0 \tag{4}$$

But from equation (4), assuming that the roots are real and distinct, we separate the even power terms from those of odd power. Thus putting $n=1$, in the above equation (4), we have;

$$ax^3 + bx^2 + cx + d = 0 \qquad (5)$$

Separating the even power terms from those of the odd power gives;

$$(ax^3 + cx) = (-bx^2 - d)$$

Squaring both sides of the above gives;

$$(ax^3 + cx)^2 = (-bx^2 - d)^2$$

$$(ax^3 + cx)(ax^3 + cx) = (-bx^2 - d)(-bx^2 - d)$$

$$a^2x^6 + acx^4 + acx^4 + c^2x^2 = b^2x^4 + bdx^2 + bdx^2 + d^2$$

$$a^2x^6 + 2acx^4 + c^2x^2 = b^2x^4 + 2bdx^2 + d^2$$

Collecting the like terms

$$a^2x^6 + 2acx^4 + c^2x^2 - b^2x^4 - 2bdx^2 - d^2 = 0$$

$$a^2x^6 + 2acx^4 - b^2x^4 + c^2x^2 - 2bdx^2 - d^2 = 0 \qquad (6)$$

Putting $x^2 = y$ in equation (6), gives;

$$a^2y^3 + 2acy^2 - b^2y^2 + c^2y - 2bdy - d^2 = 0$$

$$a^2y^3 + (2acy^2 - b^2)y^2 + (c^2y - 2bd)y - d^2 = 0$$

Dividing through by a^2 in the above, we have;

$$y^3 + \frac{(2acy^2 - b^2)y^2}{a^2} + \frac{(c^2 - 2bd)y}{a^2} - \frac{d^2}{a^2} = 0 \qquad (7)$$

From (7), let

$$P_1 = \frac{2ac - b^2}{a^2},\ P_2 = \frac{c^2 - 2bd}{2ac - b^2}, \text{ and } P_3 = \frac{d^2}{c^2 - 2bd}$$

Hence the roots of the original equation as given in equation (7) are;

$$x_i = \pm\sqrt[2^r]{P_i}$$

And substituting the initial coefficients from equation (5) into (7) gives;

$$\Rightarrow x_1 = \pm\sqrt[2]{\left|\frac{2ac-b^2}{a^2}\right|}\ ;\ x_2 = \pm\sqrt[2]{\left|\frac{c^2-2bd}{2ac-b^2}\right|} \text{ and } x_3 = \pm\sqrt[2]{\left|\frac{d^2}{c^2 2bd}\right|}$$

Second Iteration
From (7) let

$$a^2y^3 + 2acy^2 - b^2y^2 + c^2y - 2bdy - d^2 = 0 \tag{8}$$

$$a^2y^3 + (2ac-b^2)y^2 + (c^2-2bd)y - d^2 = 0$$

Let $E = a^2,\ F = 2ac-b^2,\ G = c^2-2bd,\ H = d^2$

$$Ey^3 + Fy^2 + Gy - H = 0$$

Separating the even power terms from those of the odd power gives;

$$(Ey^3 + Gy) = (-Fy^2 + H)$$

Squaring both sides of the above gives;

$$(Ey^3 + Gy)^2 = (-Fy^2 + H)^2$$

$$(Ey^3 + Gy)(Ey^3 + Gy) = (-Fy^2 + H)(-Fy^2 + H)$$

$$E^2y^6 + EGy^4 + EGy^4 + G^2y^2 = F^2y^4 - FHy^2 - FHy^2 + H^2$$

$$E^2y^6 + 2EGy^4 + G^2y^2 = F^2y^4 - 2FHy^2 + H^2$$

Collecting the like terms

$$E^2y^6 + 2EGy^4 - F^2y^4 + G^2y^2 + 2FHy^2 - H^2 = 0$$

$$E^2y^6 + (2EG-F^2)y^4 + (G^2+2FH)y^2 - H^2 = 0 \tag{9}$$

Putting $y^2 = z$ in equation (9) gives;

$$E^2z^3 + (2EG-F^2)z^2 + (G^2+2FH)z - H^2 = 0$$

Dividing through by E^2 in the above, we have;

$$Z^3 + \frac{(2EG^2-F^2)z^2}{E^2} + \frac{(G^2+2FH)z}{E^2} - \frac{H^2}{E^2} = 0 \tag{10}$$

From (10), let

$$P_1 = \frac{2EG - F^2}{E^2},\ P_2 = \frac{G^2 + 2FH}{2EG - F^2}, \text{ and } P_3 = \frac{H^2}{G^2 + 2FH}$$

Hence the roots of the original equation as given in (10) are;

$$x_i = \pm\sqrt[2^r]{P_i}$$

And substituting the initial coefficients from equation (5) into (10) gives;

$$x_1 = \pm\sqrt[4]{\frac{2EG - F^2}{E^2}} \quad \Rightarrow x_1 = \pm\sqrt[4]{\left|\frac{2a^2(c^2 - 2bd) - (2ac - b^2)^2}{a^4}\right|}$$

$$x_2 = \pm\sqrt[4]{\frac{G^2 + 2FH}{2EG - F^2}} \quad \Rightarrow x_2 = \pm\sqrt[4]{\left|\frac{(c^2 - 2bd)^2 + 2[2ac - b^2]d^2}{2a^2(c^2 - 2bd) - (2ac - b^2)^2}\right|}$$

$$x_3 = \pm\sqrt[4]{\frac{H^2}{G^2 + 2FH}} \quad \Rightarrow x_3 = \pm\sqrt[4]{\left|\frac{(d^2)^2}{(c^2 - 2bd)^2 + 2[2ac - b^2]d^2}\right|}$$

Third Iteration

From (10) let

$$E^2 z^3 + (2EG - F^2)z^2 + (G^2 + 2FH)z - H^2 = 0 \tag{11}$$

Let $D = E^2,\ P = 2EG - F^2,\ Q = G^2 + 2FH, W = H^2$

$$Dz^3 + Pz^2 + Qz - W = 0$$

Separating the even power terms from those of the odd power gives;

$$(Dz^3 + Qz) = (-Pz^2 + W)$$

Squaring both sides of the above gives;

$$(Dz^3 + Qz)^2 = (-Pz^2 + W)^2$$

$$(Dz^3 + Qz)(Dz^3 + Qz) = (-Pz^2 + W)(-Pz^2 + W)$$

$$D^2 z^6 + DQz^4 + DQz^4 + Q^2 z^2 = P^2 z^4 - PWz^2 - PWz^2 + W^2$$

$$D^2 z^6 + 2DQz^4 + Q^2 z^2 = P^2 z^4 - 2PWz^2 + W^2$$

Collecting the like terms

$$D^2 z^6 + 2DQz^4 - P^2 z^4 + Q^2 z^2 + 2PWz^2 - W^2 = 0$$

$$D^2 z^6 + (2DQ - P^2) z^4 + (Q^2 + 2PW) z^2 - W^2 = 0 \tag{12}$$

Putting $z^2 = L$ in equation (12) gives;

$$D^2 L^3 + (2DQ - P^2) L^2 + (Q^2 + 2PW) L - W^2 = 0$$

Dividing through by D^2 in the above, we have;

$$L^3 + \frac{(2DQ^2 - P^2)L^2}{D^2} + \frac{(Q^2 + 2PW)L}{D^2} - \frac{W^2}{D^2} = 0 \tag{13}$$

From (13), let

$$P_1 = \frac{2DQ - P^2}{D^2}, \; P_2 = \frac{Q^2 + 2PW}{2DQ - P^2}, \text{ and } P_3 = \frac{W^2}{Q^2 + 2PW}$$

Hence the roots of the original equation as given in equation (13), are;

$$x_i = \pm \sqrt[2^r]{P_i}$$

And substituting the initial coefficients from equation (5) into (13) gives;

$$x_1 = \pm \sqrt[8]{\frac{2DQ - P^2}{D^2}} \quad x_1 = \pm \sqrt[8]{\left| \frac{2a^4 \left[(c^2 - 2bd)^2 + 2(2ac - b^2)d^2 \right] - \left[2a^2 (c^2 - 2bd) - (2ac - b^2)^2 \right]}{a^4} \right|}$$

$$x_2 = \pm \sqrt[8]{\frac{Q^2 + 2PW}{2DQ - P^2}}$$

$$x_2 = \pm \sqrt[8]{\left| \frac{\left(c^2 - 2bd\right)^2 + 2\left(2ac - b^2\right)d^2 + \left[2\left(2a^2 (c^2 - 2bd)\right)\left(2ac - b^2\right)^2 \right] d^4}{2a^4 \left[(c^2 - 2bd)^2 + 2(2ac - b^2)d^2 \right] - \left[2a^2 (c^2 - 2bd) - (2ac - b^2)^2 \right]} \right|}$$

$$x_3 = \pm \sqrt[8]{\frac{W^2}{Q^2 + 2PW}}$$

$$x_3 = \pm \sqrt[8]{\left| \frac{\left(d^4\right)^2}{\left[\left(c^2 - 2bd\right)^2 + 2\left(2ac - b^2\right)d^2 \right]^2 - \left[2\left(2a^2 (c^2 - 2bd)\right) - \left(2ac - b^2\right)^2 \right] d^4} \right|}$$

Fourth Iteration:
From (13) let

$$D^2L^3 + (2DQ - P^2)L^2 + (Q^2 + 2PW)L - W^2 = 0 \quad (14)$$

Let $T = D^2,\ R = 2DQ - P^2,\ N = Q^2 + 2PW,\ M = W^2$

$$TL^3 + RL^2 + NL - M = 0$$

Separating the even power terms from those of the odd power gives;

$$(TL^3 + NL) = (-RL^2 + M)$$

Squaring both sides of the above, gives;

$$(TL^3 + NL)^2 = (-RL^2 + M)^2$$

$$(TL^3 + NL)(TL^3 + NL) = (-RL^2 + M)(-RL^2 + M)$$

$$T^2L^6 + TNL^4 + TNL^4 + N^2L^2 = R^2L^4 - RML^2 - RML^2 + M^2$$

$$T^2L^6 + 2TNL^4 + N^2L^2 = R^2L^4 - 2RML^2 + M^2$$

Collecting the like terms

$$T^2L^6 + 2TNL^4 - R^2L^4 + N^2L^2 + 2RML^2 - M^2 = 0$$

$$T^2L^6 + (2TN - R^2)L^4 + (N^2 + 2RM)L^2 - M^2 = 0 \quad (15)$$

Putting $L^2 = J$ in equation (15), gives;

$$T^2J^3 + (2TN - R^2)J^2 + (N^2 + 2RM)J - M^2 = 0$$

Dividing through by T^2 in the above, we have;

$$J^3 + \frac{(2TN - R^2)J^2}{T^2} + \frac{(N^2 + 2RM)J}{T^2} - \frac{M^2}{T^2} = 0 \quad (16)$$

From (3.25), let

$$P_1 = \frac{2TN - R^2}{T^2},\ P_2 = \frac{N^2 + 2RM}{2TN - R^2},\ \text{and } P_3 = \frac{M^2}{N^2 + 2RM}$$

Hence the roots of the original equation as given in equation (16), are;

$$x_i = \pm \sqrt[2^r]{P_i}$$

And substituting the initial coefficients from equation (5) into (16) gives;

$$x_1 = \pm\sqrt[16]{\frac{2TN - R^2}{T^2}} \quad ,$$

$$x_1 = \pm\sqrt[8]{\frac{2a^8\left[\left[\left(c^2-2bd\right)^2+2\left(2ac-b^2\right)d^2\right]^2+2\left[2a^2\left(c^2-2bd\right)-\left(2ac-b^2\right)^2\right]d^4\right] - \left[2a^4\left[\left(c^2-2bd\right)^2+2\left(2ac-b^2\right)d^2\right]-\left[2a^2\left(c^2-2bd\right)-\left(2ac-b^2\right)^2\right]^2\right]^2}{\left[a^2\right]^8}} \tag{17}$$

$$x_2 = \pm\sqrt[16]{\frac{N^2 + 2RM}{2TN - R^2}}$$

$$x_2 = \pm\sqrt[16]{\frac{\left[\left[\left(c^2-2bd\right)^2+2\left(2a^2c-b^2\right)d^2\right]^2+2\left[2a^2\left(c^2-2bd\right)-\left(2a^2c-b^2\right)^2\right]d^4\right]^2 - 2\left[2a^4\left[\left(c^2-2bd\right)^2-2\left(2a^2c-b^2\right)d^2\right]-\left[2a^2\left(c^2-2bd\right)-\left(2a^2c-b^2\right)^2\right]^2\right]d^8}{2a^8\left[\left[\left(c^2-2bd\right)^2+2\left(2ac-b^2\right)d^2\right]^2+2\left[2a^2\left(c^2-2bd\right)-\left(2ac-b^2\right)^2\right]d^4\right] - \left[2a^4\left[\left(c^2-2bd\right)^2+2\left(2ac-b^2\right)\right]-\left[2a^2\left(c^2-2bd\right)-\left(2ac-b^2\right)^2\right]^2\right]^2}} \tag{18}$$

$$x_3 = \pm\sqrt[16]{\frac{M^2}{N^2 + 2RM}}$$

$$X_3 = \pm\sqrt[16]{\frac{\left(d^2\right)^8}{\left[\left[\left(c^2-2bd\right)^2+2\left(2a^2c-b^2\right)d^2\right]^2+2\left[2a^2\left(c^2-2bd\right)-\left(2a^2c-b^2\right)^2\right]d^4\right] - 2\left[2a^4\left[\left(c^2-2bd\right)^2-2\left(2a^2c-b^2\right)d^2\right]-\left[2a^2\left(c^2-2bd\right)-\left(2a^2c-b^2\right)^2\right]^2\right]d^8}} \tag{19}$$

APPLICATION OF THE SCHEME TO THE THIRD ORDER POLYNOMIAL PROBLEM SOLUTIONS

Example 3.1

Obtain the solution to the polynomial $x^3+2x^2-5x-6=0$

Table 1: Comparison between Exact solution and Graeffe method via Tabular and Graphical Profiles

Iteration	Exact Method			Graeffe Method			Error		
	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
1	-3.000000000	2.000000000	-1.000000000	-3.741657387	1.870828693	-0.857142856	0.741657387	0.129171307	-0.142857144
2	-3.000000000	2.000000000	-1.000000000	-3.146346284	1.941696108	-0.982117548	0.146346284	0.058303892	-0.017882452
3	-3.000000000	2.000000000	-1.000000000	-3.014443336	1.991425261	-0.999493821	0.014443336	0.008574739	-0.000506179
4	-3.000000000	2.000000000	-1.000000000	-3.000285258	1.999811756	-0.999999044	0.000285258	0.000882440	-0.000000956

Table 2: Extended version of table 1

	Exact values		Graeffe values	
S/N	x	y	x	y
1	-4.000000000	-18.00000000	-4	-18
2	-3.000000000	0.000000000	-3.000285258	-0.00285315
3	-2.000000000	4.000000000	-3.999999999	-17.99999997
4	-1.000000000	0.000000000	-2.999999999	1E-08000000
5	0.000000000	-6.000000000	-1.999811756	3.999811614
6	1.000000000	-8.000000000	-1.999999999	3.999999999
7	2.000000000	0.000000000	-0.999999044	-5.736E-06000
8	3.000000000	24.00000000	-0.99999999	-6E-08000000
9	4.000000000	70.00000000	0	-6

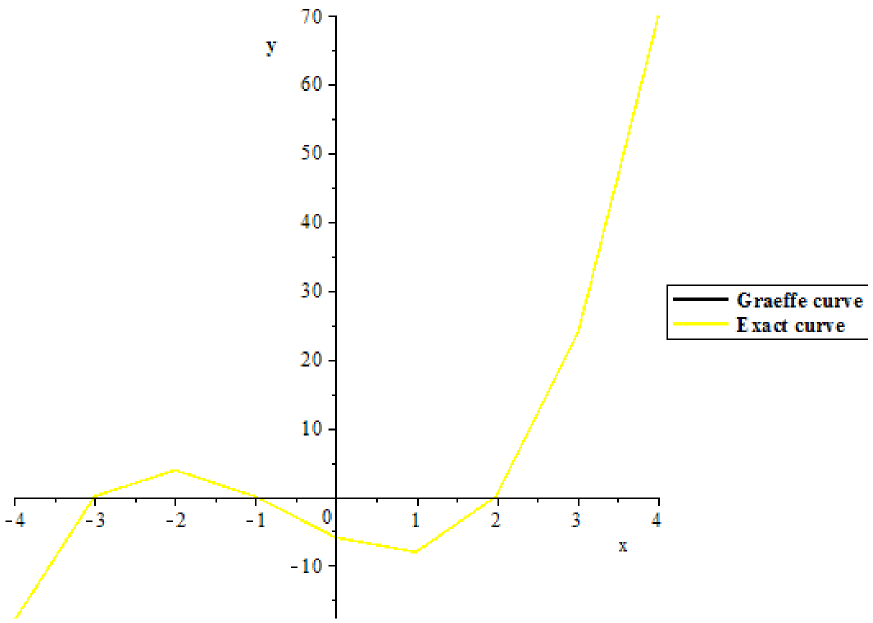

Fig 1: Graphical Comparison of Exact and Graeffe's Solution for $x^3+2x^2-5x-6=0$

Example 3. 2

Obtain the solution to the polynomial $x^3 - x^2 - 4x + 4 = 0$

Table 3: Comparison between Exact solution and Graeffe method via Tabular and Graphical Profiles

Iteration	Exact Method			Graeffe Method			Error		
	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
1	2.000000000	-2.000000000	1.000000000	3.000000000	-1.632993162	0.81649658	-1.000000000	-0.367006838	0.183503420
2	2.000000000	-2.000000000	1.000000000	2.396781727	-1.71877741	0.970983543	-0.396781727	-0.281222590	0.029012457
3	2.000000000	-2.000000000	1.000000000	2.181547485	-1.835345318	0.999027705	-0.181547485	-0.164654682	0.000972295
4	2.000000000	-2.000000000	1.000000000	2.088548561	-1.915209301	0.999998092	-0.088548500	-0.047906990	0.000001908

Table 4: Extended version of table 3

Number of iterations	Exact values		Graeffe values	
S/N	x	y	x	y
1	-3.000000000	-20.00000000	-3.000000000	-20.00000000
1	-2.000000000	0.000000000	-2.0885485610	-1.118162961
1	-1.000000000	6.000000000	-2.9999999990	-19.99999997
2	0.000000000	4.000000000	-1.9152093010	0.967771749
2	1.000000000	0.000000000	-1.9999999990	1.2E-0800000
2	2.000000000	0.000000000	-0.9999999990	6.000000001
3	3.000000000	10.00000000	0.000000000	4.000000000
3	4.000000000	36.00000000	0.9999980920	5.72401E-06
3	5.000000000	84.00000000	0.9999999990	3E-09000000

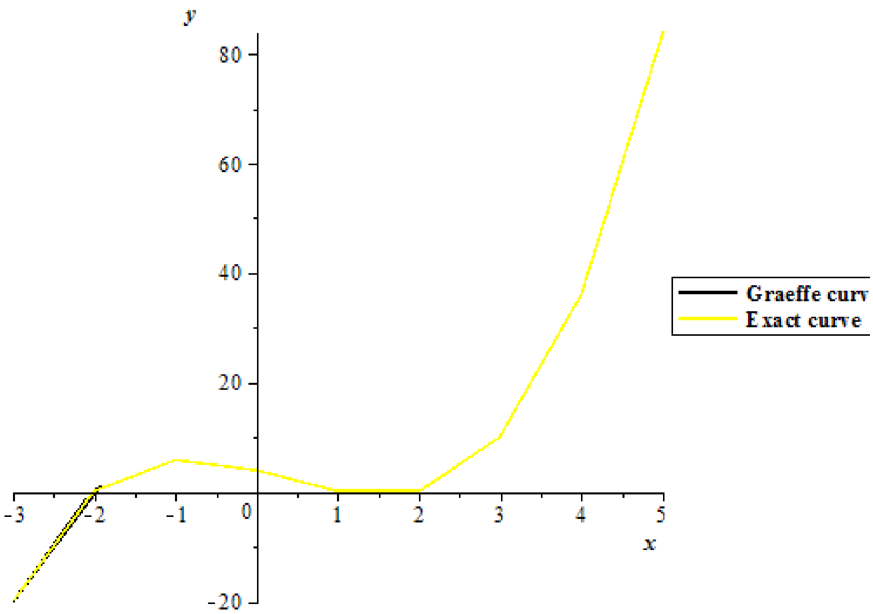

Fig 2: Graphical Comparison of Exact and Graeffe's Solution for $x^3 - x^2 - 4x + 4 = 0$

APPLICATION AND COMPARISON OF THE GRAEFFE'S SCHEME AND THE NEWTON RAPHSON METHOD

Example 3.3

Obtain the solution to polynomial $x^2 - 21 = 0$ $\quad$ *by* Patil and Verma (2009)

Note

For a second-degree problem, using equation (5) we set the coefficient a = 0 in order to use solution results in equations (17), (180 and (19) respectively.

$$i.e.; \quad 0(x^3) + bx^2 + cx + d = 0$$

Hence, the table 5 below comprises the solution results to example 3.3 via Newton Raphson and Graeffe's method.

Table 5: Comparison between the Results of Newton Raphson and Graeffe method.

Iteration	Newton Raphson by Patil and Verma (2009)		Graeffe method		Exact	
	x_1	x_2	x_1	x_2	x_1	x_2
1	6.250000000	--	6.480740698	-3.240370349	4.582575695	- 4.582575695
2	5.787600000	--	5.449631621	-3.853471475	4.582575695	- 4.582575695
3	5.414534658	--	5.732955275	-3.663032240	4.582575695	- 4.582575695
4	5.130838069	--	4.735463654	-4.388289518	4.582575695	- 4.582575695

Example 3.2

Obtain the solution to the polynomial $x^3 - 2x - 5 = 0$ $\quad$ Sastry (2012)

Table 6: Comparison between the Results of Newton Raphson and Graeffe method.

Iteration	Newton Raphson Method by Sastry (2012)			Graeffe Method			Error		
	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
1	2.100000000	--	--	2.000000000	-1.000000000	2.500000000	2.094551482	Complex Root	Complex Root
2	2.094568000	--	--	1.681792831	-2.189938703	-1.357580367	2.094551482	Complex Root	Complex Root
3	2.094551000	--	--	2.135184796	1.651065420	1.418307179	2.094551482	Complex Root	Complex Root
4	2.094551000	--	--	2.096144584	-1.589922474	-1.500281731	2.094551482	Complex Root	Complex Root
5	2.094551000	--	--	2.094551082	-1.019922474	-1.011922474	2.094551482	Complex Root	Complex Root

DISCUSSION OF RESULTS

From table 1, 2, 3, 4 and the graphs in figures 1 and 2, it could be deduced that our proposed Graeffe's scheme compared favourably with the exact solution of the problem considered. Another insight from the graphs of the solution to the problems plotted suggested that the Graeffe's method of solution coincided with the exact solution after the fourth iteration, which thus recommends the method to be suitable for the solutions of such family of problems considered.

Similarly, from tables 5 and 6 using the work of [11], it could be observed that after the fifth iteration, the Graeffe's method converges approximately to the exact solution. And thus, it could be deduced that our scheme competed favourably with the existing method by Newton Raphson and even in most cases noted to be much better (when the comparison analysis was made) using the works of Patil and Verma (2009) and Sastry (2012) respectively. Moreso, for problems with complex roots solved using table 6, while Newton Raphson's method could not give further information about the complex roots, the Graeffe's method gave values that almost coincided with the real parts of the complex roots of the problem considered.

CONCLUSION

According to [2, 10, 11], it is known that every polynomial of degree n must have n-number of roots. Thus, the comparison analysis made by this study with respect to the exact solutions of the considered problems revealed that using the Graeffe's method as derived in our scheme will always succeed in providing all the n-number of roots as required for any polynomial of degree n unlike using the Newton Method. Specifically, in some cases where it is difficult to solve the polynomial using the exact/analytical method, the Graeffe's method developed in this study will prevail and become a better option provided the roots are real or distinct.

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