Journal of Research in Engineering and Computer Sciences
April 2024, Vol. 2, No. 2, pp. 14-19

On the Refined Neutrosophic Nonabelian P-Groups of A Given Order

Sunday Adesina Adebisi

1. Department of Mathematics, Faculty of Science, University of Lagos, Akoka, Yaba, Nigeria

Abstract:
In every nonabelian neutrosophic p-group G(I), possessing two neutrosophic cyclic subgroups X(I)i and X(I)j, the quotient neutrosophic group of G (I) by X(I)i is isomorphic to the neutrosophic cyclic group X(I)j for i, j ∈ {1, 2}, i /= j. Moreover, if p > 2 and G(I) is metacyclic, possessing a neutrosophic nonabelian section Y (I), of order p3, then Y (I) is a trivial neutrosophic subgroup of G(I).

Keywords: Nonabelian neutrosophic group, neutrosophic quotient group, neutrosophic cyclic group, neutro- sophic section, neutrosophic pyramidal group

INTRODUCTION

The finite neutrosophic p-groups possess many remarkable characterizations of which majority of their proofs were given by Burnside, Frobenius, Sylow and a host of other Mathematicians.

This paper purposes to present some of the input efforts in classifying the neutrosophic p-groups most especially, those with a neutrosophic cyclic subgroup of index p. Part of the work also leads to the computation of the number of neutrosophic subgroups of a given order in a neutrosophic metacyclic p-group. A lot of developments are ongoing concerning the concepts of refined neutrosophic algebraic structures. Neutrosophic researchers such as Agboola Adesina has been able to successfully introduce vital and useful tools in this particular regard (please, see [2]). After the suc- cessful feat, many other neutrosophic researchers have as well tried to establish and studied further more on the refined neutrosophic algebraic structures. (Please, see [3]). Further studies on refined neutrosophic rings and refined neutrosophic subrings, their presentations and fundamental were also worked upon.

Also, Agboola, in his paper [1] has examined and as well studied the refined neutrosophic quotient groups, where more properties of refined neutrosophic groups were presented and it was shown that the classical isomorphism theorems of groups do not hold in the refined neu- trosophic groups. The existence of classical morphisms between refined neutrosophic groups G $(I_1; I_2)$ and neutrosophic groups G(I) were established. The readers can as well consult [4, 5, 6, 7 and 13] in order to have detailed knowledge concerning the refined neutrosophic logic, neutrosophic groups, refined neutrosophic groups and neutrosophy, in general.

Please note the following:
Throughout this paper, our BINARY OPERATION is strictly not of multiplication (as this may not be defined due to the fact that I^{-1} does not exist)

Definition 1
(Please, see [1]): Suppose that $(X(I_1; I_2); +; .)$ is any refined neutrosophic algebraic structure. Here, + and. are ordinary addition and multiplication respectively. Then I_1 and I_2 are the split

components of the indeterminacy factor I that is $I = \alpha_1 I_1 + \alpha_2 I_2$ with $\alpha_i \in C; i = 1; 2$.

Definition 2

(Please, see [1]): Suppose that (G; ∗) is any group. Then, the couple $(G(I_1; I_2); *)$ can be referred to as the refined neutrosophic group. Furthermore, this group can be said to be generated by G, I_1 and I_2 and $(G(I_1; I_2); *)$ is said to be commutative if $\forall x; y \in G(I_1; I_2)$, we have $x * y = y * x$. Otherwise, $(G(I_1; I_2); *)$ can be referred to as a non- commutative refined neutrosophic group.

Theorem 1

(Please, see [1]): (1) Every refined neutrosophic group is a semigroup but not a group. (2) Every refined neutrosophic group contains a group.

Corollary

(Please, see [1]): Every refined neutrosophic group $(G(I_1; I_2); +)$ is a group.

Definition 3

(Please, see [1]): Let $(G(I_1; I_2); *)$ be a refined neutrosophic group and let $A(I_1; I_2)$ be a nonempty subset of $G(I_1; I_2)$. $A(I_1; I_2)$ is called a refined neutrosophic sub-group of $G(I_1; I_2)$ if $(A(I_1; I_2); *)$ is a refined neutrosophic group. It is essential that $A(I_1; I_2)$ contains a proper subset which is a group. Otherwise, $A(I_1; I_2)$ will be called a pseudo refined neutrosophic subgroup of $G(I_1; I_2)$.

Definition 4

(Please, see [1]): Let $H(I_1; I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1; I_2);)$. Define $x = (a; bI_1; cI_2) \in G(I_1; I_2)$.

PRELIMINARIES AND ESSENTIAL DEFINITIONS

- $B(I)^x = x^{-1}B(I)x = \{b^x | b \in B(I)\}$ for $x \in G(I)$, $B(I) \subseteq G(I)$.
- $[a, b] = a^{-1}b^{-1}ab = a^{-1}a^b$. The commutator of elements a and b of a group $G(I)$.
- $|W|$ is the cardinality of the set W. If $G(I)$ is a finite group, then $|G(I)|$ is called the order of $G(I)$.
- o(x) is the order of an element x of G(I).
- $cl(G(I))$ is the nilpotence class of aneutrosophic p-group $G(I)$. Here, there exists a series given by: $G(I) = G(I)_0 \geq G(I)_1 \geq \cdots \geq G_n \geq G(I)_{n+1} = \{e\}$. And we say that G(I) is of class n. We then write $cl(G(I)) = n > 1$.
- A neutrosophic p-group of maximal class is a neutrosophic nonabelian group $G(I)$ oforder p^m, $m \geq 3$ with $cl(G(I)) = m - 1 > 1$.
- Let $G(I) = G(I)_1 \times \cdots \times G(I)_n$, $A \leq G(I)$ and $a \in A$. Then $a = (a_1, \ldots, a_n)$, where $a_i \in G(I)_i \; \forall \; i$. Define a projection $\pi_i: A \longrightarrow G(I)_i$, setting $\pi_i(a) = a_i$, all $a \in A$. Then, π_i is a homomorphism and $A_i = \pi_i(A)$ is an epimorphic image of A. This is called the section of A. Obviously, $A \leq A_1 \times \cdots \times A_n$.
- A section of a neutrosophic group $G(I)$ is an epimorphic image of some neutrosophic subgroups of $G(I)$.
- $\Omega_n(G(I)) = < x \in G(I) | \, o(x) \leq p^n >$
- $\mho_n(G(I)) = < x^p \; | x \in G(I) >$
- $A \lhd B \Longrightarrow A$ is a nontrivial normal neutrosophic subgroup of $G(I)$, where $H > \{1\}$.
- A neutrosophic p-group $G(I)$ is said to be homocyclic if it is of type$(p^n, p^n, \ldots, p^n)$, $n > 0$.

- Metacyclic neutrosophic Group: This is a group G(I) in which both its commutator subgroup $G(I)^{/}$ and the quotient group $G(I)/G(I)^{/}$ are cyclic. It has a neutrosophic cyclic normal subgroup (I)L such that G(I)/L(I) is also cyclic.

PROOF OF THE RESULTS

Definition

Let G(I) be a group. If G(I) is nonabelian but all its proper subgroups are abelian, then G(I) is said to be minimal nonabelian.

Proposition 1: [8]

Suppose that G(I) is a neutrosophic metacyclic p-group containing a neutrosophic nonabelian subgroup V (I) of order p^3. Then

i. if $p = 2$, $G(I)$ is of maximal class
ii. if $p > 2$, then $|G(I)| = p^3 \Rightarrow G(I) = V(I)$.

Lemma 2 (see [8] [12])

Suppose that G(I) is a neutrosophic nonabelian metacyclic p-group.

i. If $G(I)$ is of order p^4 and exponent p^2 then $G(I)$ is minimal nonabelian. Moreso, if $p = 2$, then $G(I)$ is isomorphic to the group

$$H_2 = \langle x, y | x^4 = y^4 = 1,\ x^y = x^3 \rangle,$$

where all subgroups of order 2 are characteristic in $G(I)$. All maximal neutrosophic cyclic subgroups of $G(I)$ have order p^2.

ii. By Proposition 1. If $G(I)$ has a neutrosophic nonabelian subgroup of order p^3, then it is of maximal class. To be specific, if $p > 2$, then $|G(I)| = p^3$.
iii. Let B(I) be a normal neutrosophic subgroup of *G(I)* and $p = 2$ such that $G(I)/B(I)$is nonabelian of order 2^3, then *B(I)* is characteristic in *G(I).*
iv. By Theorem 1 and Lemma 2, there are exactly four series of nonabelian 2-groups with neutrosophic cyclic subgroup of index 2: viz: R_2n, D_2n, Q_2n, and SD_2n.

Lemma 3

Suppose that G(I) is a nonabelian 2-group and $|G(I):\ G(I)^{/}| = 4$. Then G(I) is as in theorem 1 (ii(a), (b), (c)).

Lemma 4

[8] Let G(I) be a metacyclic 2-group with a nonabelian section of order 2^3. If G(I) is not of maximal class, then

i. There exists a normal neutrosophic subgroup $T(I) \lhd G(I)$ such that $G(I)/T(I)$ is isomorphic to $Q(I)_{2}3$ and $G(I)/\mho_2(G(I))$ is isomorphic to $H(I)_2$.
ii. $V_1(G(I))$ has no nonabelian section of order 2^3.

Here comes the analysis of the first part of the assertion.

Let G(I) be as in theorem 1.

If $B(I)_i \subseteq G(I)$, $i \in \{1, 2\}$, such that $B(I)_i$ is cyclic, we assert that

$$|G(I)/B(I)_i| = |B(I)_j|,\ i \neq j$$

In doing this, it suffices to show that G(I)/B(I)i is isomorphic to B(I)j, i /= j, i, j ∈ {1, 2}. Define a mapping as follows

$$f : (G(I)/B(I)_i) \longrightarrow B(I)_j\ i \neq j$$
$$: xB(I)_i \rightarrowtail y\ (*)$$

We show that (∗) is (i) a monomorphism, (ii) an epimorphism.

Suppose that $|B(I)_i| < |B(I)_j|$, $i \neq j$. Then for all $x \in G(I)$, there exists $y \in B(I)_j$ such that $f(xB(I)_i) = y$.

Also, if $f(x_1B(I)_i) = f(x_2B(I)_i) = y$.Then,

$$x_1B(I)_i = x_2B(I)_i \qquad (2)$$

And by cancellation law (see [9], [11]) post multiply both sides of (2) by $B(I)_i^{-1}$ we have that

$$x_1B(I)_iB(I)_i^{-1} = x_2B(I)_iB(I)_i^{-1}$$

$$\Rightarrow x_1 = x_2.$$

This confirms (i) and (ii) as stated above.

The case is similar for $|B(I)_i| > |B(I)_j|$, $i \neq j$, $i, j \in \{1, 2\}$

In dealing with the second aspect of the problem, the following items are very imperative: Define the upper Ω-series:

$$\{1\} = \Omega_{(0)}(G(I)) < \Omega_{(1)}(G(I)) < \cdots < \Omega_{(s)}(G(I)) < \cdots$$

of a neutrosophic p-group $G(I)$ as follows:

$$\Omega_{(0)}(G(I)) = \{1\}$$

$$\Omega_{(i+1)}(G(I))/\Omega_{(i)}(G(I)) = \Omega_1(P/\Omega_{(i)}(G(I))),\ \ i = 0, 1, \ldots$$

Clearly, $\Omega_i(G(I)) \leq \Omega_{(i)}(G(I))$.

$\Omega_{(i)}(G(I)) = \Omega_{(i+1)}(G(I))$ implies $\Omega_{(i)}(G(I)) = G(I)$.

But $\Omega_{(i)}(G(I)) = \Omega_{(i+1)}(G(I))$ does not imply that $\Omega_i(G(I)) = G(I)$. Now, set

$$.\ \Omega_{(i+1)}(G(I)) : \Omega_{(i)}(G(I)) = p^{t_{i+1}}$$

$$|\mho_i(G(I)) : \mho_{i+1}(G(I))| = p^{v_{i+1}}, i = 0, 1, \ldots$$

Definition A_1 [8].

i. $G(I)$ is said to be upper pyramidal if $t_1 \geq t_2 \geq \cdots$
ii. $G(I)$ is said to be lower pyramidal if $v_1 \geq v_2 \geq \cdots$.

Definition A_2

A neutrosophic p-group $G(I)$ is said to be generalized homocyclic if it satisfies the following conditions as noted in definition A_1.

i. $t_1 = t_2 = \cdots = v_1 = v_2 = \cdots$
ii. $\Omega(i+1)(G(I))/\Omega(i)(G(I))$ are abelian for all nonnegative integer i.
iii. If D(I) is a term of the upper or lower central series of G(I), then $D(I) = \Omega_i(G(I))$ for some nonnegative integer i.

If G(I) is a generalized homocylic neutrosophic group of exponent p^e, then $\Omega_{(i)}(G(I)) = \Omega_i(G(I))$ and $\exp(\Omega_i(G(I))) = p^i \ \forall \ i \leq e$ [8].

Now, suppose that a metacyclic p-group, $p > 2$ has a nonabelian neutrosophic section $Y(I)$, of order p^3. If $p = 2$ and $G(I)$ is not of maximal class, then the result is in harmony with Lemma 4 (i) and (ii).

On the other hand, if p > 2, then going by induction on |G(I)|, assuming G(I) is non- abelian. Then, by Lemma 3, if G(I) is neither cyclic nor a 2-group of maximal class, then

$$\Omega_1(G(I)) \cong Ep^2.$$

By proposition 1, $|G(I)| = p^2$.

$\Longrightarrow G = Y$ (its section)

$\Longrightarrow Y$ is a trivial subgroup of $G(I)$.

If $G(I)$ is nonabelian of order p^4 and exponent p^2, it is not generalized homocyclic (see section 8 of [8]). (We have that $t_1 = t_2 = v_1 = v_2 = 2$),

If $p > 2$, then $G(I)$ is metacyclic. So, $|G(I)/G(I)'| = p^3$ and $G(I)' \neq \Omega_1(G(I))$

Funding:

This research received no external funding

ACKNOWLEDGMENTS

The author is very grateful to the anonymous reviewers for their helpful comments and corrections which have improved the overall quality of the work.

Conflicts of Interest

The author declares that there is no competing of interests

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