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New Formula for Calculations of Scattering Cross-Sections According to Quantum Mechanics

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Abstract:
We have calculated the general elastic and inelastic scattering differential and total cross-sections using a simple general solution of radial Schrödinger equation for a particle or particle current in a central field, generally. Before, we have obtained the general simple formulas of the scattering amplitudes and elastic, inelastic (no-elastic) and total scattering cross-sections. After, we have made some applications, numerically. In these applications, the barrier potentials are assumed to have centrifugal plus Coulomb potentials. With these potentials, the elastic, inelastic (absorption, radiation, particle detachment, particle capture, etc.) and total scattering cross-sections of different targets were calculated. Scattering cross-sections for ${}_{2}^{3}\mathrm{He}$ particles of three different energies on many targets, from ${}_{4}^{9}\mathrm{Be}$ to ${}_{82}^{208}\mathrm{Pb}$, are calculated. The calculated results are compared with experimental results. The results calculated have given satisfactory agreement with the available experimental results. Then, when the experimental elastic and reaction cross sections are known, we calculated the width of the potential barrier and the coefficient of transmission of potential barrier. By taking thermal neutrons as incoming particles, the potential barrier transmission coefficients and barrier widths were calculated for many nuclei as examples.

Keywords: Cross-sections, elastic scattering, inelastic (no-elastic), scattering theory, transmission coefficient

INTRODUCTION

When a flowing particle or particle current is encountered with a potential energy barrier greater than its total energy, it cannot pass the potential barrier and return to its environment or disappear within the potential according to classical physics. However, the observations indicate that such a particle or particle current may pass the potential barrier. In quantum physics, this phenomenon is called tunneling. Studying the movement of the particle or particle current, including tunneling, is called scattering. The terms elastic and inelastic scattering are used in scattering theory. These terms are also measured by scattering cross sections. Elastic or inelastic (reaction or absorption) scattering and cross-sections are generally calculated according to the partial-wave expansion method or by a semi-classical method called WKB (Wentzel-Kramers-Brillouin) in nuclear physics. In these calculations, there are approximations, and the calculations are quite difficult and complex. All these approximations were made obligatory because the Schrödinger equation could not be solved exactly. Since the exact solution of the Schrödinger equation can now be made [1, 2], so it is not necessary to make these approximations. In this working, the differential elastic, inelastic and total scattering cross-sections were calculated using a simple general method developed for the solution of the radial SE (RSE) of a particle at the central field, without making any approach to the wave functions. First, we used our solution to calculate scattering amplitudes, particle currents, and the cross-section of scatterings from a general spherically symmetric potential containing attractive and repulsive parts. Then, the scattering cross sections by several nuclei were calculated as examples. Then, the relationship between the cross sections and the tunneling or barrier transmission coefficient was calculated and made in several numerical applications. The calculated results were compared with the experimental results. All values were found to be in perfect agreement. It was then shown that if the experimentally measured cross sections are known, the coefficient of crossing the potential barrier and the width of the potential barrier can be found. Numerical applications were made.

SCATTERING THEORY

Let us consider a spherical wave progressing at the direction of Oz axis from right to left and arriving to a central potential field, sitting at the origin of the Oxyz coordinate system. When we consider scattering, we shall assume that the interaction between the scattering particle and the scatter can be represented by an effective central potential energy function U(r), where r is the relative radial variable. The effective potential U(r) can include attractive and repulsive parts. Such a central potential is schematically represented in Figure 1. The total energy of the incoming particle beam is E, and the incoming particle beam can be represented by the spherical wave. This progressive spherical wave progress from right to left and arrives to the point $\mathrm{r = r_1}$ in Figure 1. We divide the potential region into three zones and examine the motion of particle beam into these three zones.

Conversion of Potential into Two Parts

The effective potential contains both attractive (negative potential energy) and repulsive (barrier potential energy) parts. The energy of the incoming particle stream is always positive energy. To make the calculations easier and more understandable, it is more appropriate to make all the effective potential positive. If the potential is given as $\mathrm{V(r) = V_0(r) - V_{00}}$, $\mathrm{[V_0(r) > 0 \text{ and } V_{00} > 0]}$, the effective potential is $\mathrm{U(r) = V_0(r) - V_{00} + b/r^2 < 0}$ in the bound states. Here, $\mathrm{b/r^2}$ is rotational energy, and $\mathrm{-V_{00}}$ is the depth of the potential well. Let us find the maximum and minimum values of this effective potential U(r). Let the roots of the equation $\mathrm{U'(r) = 0}$, $\mathrm{r_{m1}}$ and $\mathrm{r_{m2}}$ be. $\mathrm{r_0 = r_{m1}}$ is the point where the effective potential receives the smallest values $\mathrm{U(r_{m1})}$, and $\mathrm{r_{m2}}$ is the point where the effective potential largest value $\mathrm{U(\,r_{m2}) = U_b}$. Let $\mathrm{U_0 = U(r_0) - V_{00}}$. Thus, $\mathrm{U(r) = V_0(r) - U_0 + b/r^2 < 0}$ can be written. By solving this U(r) potential directly, energy values and wave functions can be found. However, if this potential is divided into two parts, with a well and an obstacle, there may be some convenience. The obstacle comes from rotational, gravity, Coulomb, and similar energies. Therefore, the U(r) potential can be written as the sum of two parts as follows:

$$\mathrm{U(r) = U_w(r) + U_b(r)\ ;\ [\,U_w(r) = V_0(r) - U_0\ ;\ U_b(r) = b/r^2 + c/r]} \tag{1}$$

Here, the $\mathrm{U_w(r)}$ potential is the vibration part of the $\mathrm{U(r)}$ potential, and the $\mathrm{U_b(r)}$ potential is total of the rotational and the other obstacle potential parts of the potential $\mathrm{U(r)}$. Here, $\mathrm{c/r}$ is the sum of gravity, Coulomb, and similar barrier potentials. $\mathrm{U_0}$ is the depth of the potential well. If the coordinate start is taken at the point $\mathrm{(r_0\,, -U_0\,)}$, in this new coordinate system, $\mathrm{U_b(r) = b/r^2 + c/r}$ and $\mathrm{U_w(r) = V_0(r)}$. Thus, the effective potential is written as follows:

$$\mathrm{U(r) = U_w(r) + U_b(r)\ ;\ [\,U_w(r) = V_0(r)\ ;\ U_b(r) = b/r^2 + c/r]} \tag{2}$$

The graph of this potential is shown in Figure 1. (Shape of the U(r) in the coordinate system at $\mathrm{(r_0\,, -U_0)}$. In this way, three domains I, II, III are obtained. Thus, by solving the equation (2), E energy values are found in the bound states. Here these solutions do not concern us. The barrier potential $U_b(r)$ causes scattering of the wave propagating from right to left. This potential interests us here.

CALCULATION OF THE SCATTERING AMPLITUDES

Zone I is the region before the effective potential from where free particle comes; zones II, III are the effective potential regions where the particle beam is affected. These regions may include attractive and repulsive potential segments. In the scattering event, only the potential region $U_2(r) = U_b(r)$ is effective. We assume that $r = r_1$ and $r = r_2$ at the interface between zone I and II, and zone II and III, respectively. The effective potential segments in the zones are represented as $U_1(r), U_2(r)$ and $U_3(r)$ according to the zone numbers. The central potential can be taken as zero at much far from the zone I so that the particle is free in that region and the effective potential is composed of only the centrifugal term due to the incoming particle angular momentum or spin. The Coulomb interaction potential should also be added to $U_1(r)$ if that is available. The total energy of the incoming particle and the centrifugal term are always positive, and the latter is less than the former.

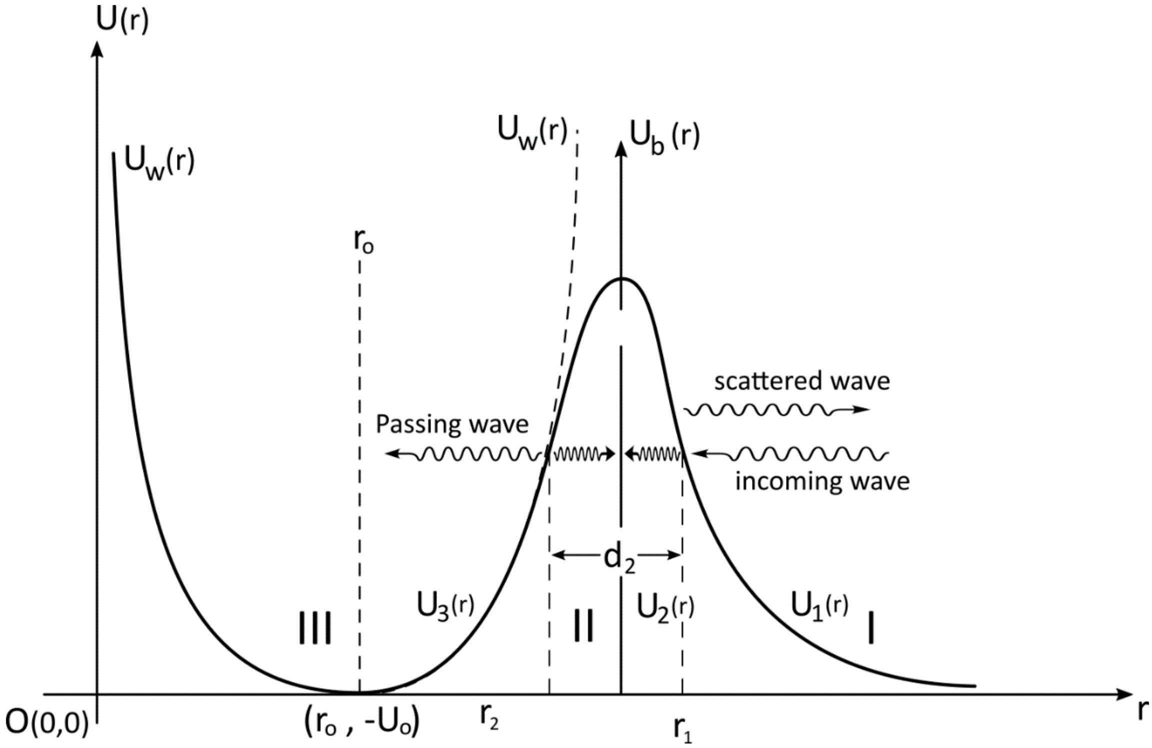

Figure 1. General schematic representation of scattering by central potential

The radial function obtained from the general solution of the RSE is given as: [1, 2]

$$R(r) = F(r)/r \ ; \ F(r) = A\, e^{k\, r \pm i\, G(r)} + B\, e^{-k\, r \mp i\, G(r)} \tag{3}$$

(a) For the case where $E > U(r)$, $k = i\, m_1 \sqrt{E}$, $G(r) = i\, m_1 \int \sqrt{U(r)}\, dr$
(b) For the case where $E < U(r)$, $k = m_1 \sqrt{E}$, $G(r) = m_1 \int \sqrt{U(r)}\, dr$

According to the functions given in Equations (3), the following functions are determined for the zones that are considered as follows:

In the zone I:
$E > 0\,, U_1(r) > 0$ and $E > U_1(r)\,;\ k = i\,m_1\sqrt{E} = i\,K\,;\ G_1(r) = i\,m_1 \int \sqrt{U_1(r)}\,dr = i\,Q_1(r)\,;$

In the zone II:
$E > 0\,,\ U_2(r) > 0$ and $E < U_2(r)\ ;\ G_2(r) = m_1 \int \sqrt{U_2(r)}\,dr = Q_2(r)\,;$

In the zone III:
$E > 0$ and $E > U_3(r)\ ;\ \ k = i\,m_1\sqrt{E} = i\,K\ ;\ G_3(r) = i\,m_1 \int \sqrt{U_3(r)}\,dr = i\,Q_3(r)\ ;$
$K = m_1\sqrt{E}\ \ ;\ \ m_1 = \sqrt{2\,m}/\hbar\ \ ;\ \ Q_p(r) = m_1 \int \sqrt{U_p(r)}\ dr\ ,\ \ (p = 1, 2, 3)$

Under these circumstances, the radial wave functions in the three zones can be put in the forms below regarding the general functions given in Equations (3).

$$F_1(r) = A_1 e^{i\,K\,r - Q_1(r)} + B_1 e^{-i\,K\,r + Q_1(r)}\ ; \tag{4a}$$

$$F_2(r) = A_2 e^{\,K\,r \pm i\,Q_2(r)} + B_2 e^{-\,K\,r \mp i\,Q_2(r)}\ ; \tag{4b}$$

$$F_3(r) = A_3 e^{-\,i\,K\,r \pm\,Q_3(r)}\ ;\ \ [\,Q_1(r) > 0\ ,\ Q_2(r) > 0\,,\ Q_3(r) > 0\,] \tag{4c}$$

The potential in zone III can also be complex in some cases (usually called the optical potential). If $U_3(r)$ is the optical potential, it can be written as follows:

$$U_3(r) = |U_3(r)| e^{i\,\emptyset} = U_{31}(r) + i\,U_{32}(r) = \sqrt{U_{31}^2(r) + U_{32}^2(r)}\ \ e^{i\,\emptyset} \tag{5a}$$

$$\tan(\emptyset) = \frac{U_{32}(r)}{U_{31}(r)}\ ,\ \ \emptyset = \arctan\left[\frac{U_{32}(r)}{U_{31}(r)}\right];\ k = i\,m_1\sqrt{E} = i\,m_1\sqrt{-|E|} = K$$

$$Q_3(r) = \int \sqrt{|U_3(r)|}\,dr = m_1 \int \sqrt{\sqrt{U_{31}^2(r) + U_{32}^2(r)}}\ dr = m_1 \int \sqrt[4]{U_{31}^2(r) + U_{32}^2(r)}\,dr \tag{5b}$$

In Equations (4a-4c), the functions $Q_p(r)$ can also be written briefly as follows:

$$Q_p(r) = m_1 \int \sqrt{\left|U_p(r)\right|}\,dr\,,\ \ [p = 0, 1, 2, 3] \tag{5c}$$

The terms of containing (A_1 and B_1) coefficients in the functions of the Equation (4a) give outgoing wave and incoming waves, respectively. We assume that the amplitude of incoming wave at the boundary of zone I and II is constant. The second (4b) and the third (4c) functions represent the states of the wave in the effective region of the potential. The (4c) function represents the wave passing from region (II) to region (III). Applying the continuity conditions on $F_p(r_j)$ and $F'_p(r_j)$, $[p, j = 1, 2, 3]$ functions, the coefficients A_1 and B_1 in Equations (4a–4c) can be determined. These conditions at the boundary points of the three zones can be written in the following form:

$$F_1(r_1) = F_2(r_1)\,;\ F'_1(r_1) = F'_2(r_1)\,;\ F_2(r_2) = F_3(r_2)\,;\ F'_2(r_2) = F'_3(r_2);$$
$$Q'_p(r_j) = K, (p\,, j = 1,2,3) \tag{6}$$

The coefficients A_1, A_2, B_2, A_3 in the functions given in Equations (4a–4c) can be found by solving four linear equations, which can be obtained by using the conditions given in Equation (6) for each of the functions given in Equations (4a–4c). These coefficients are obtained depending on the B_1 coefficient. The essential coefficients for the scattering cross section are A_1 and A_3 as described below. Therefore, there is no need to give other coefficients here. The A_1 and A_3 coefficient, which is obtained from four equations, is computed by considering the lower and upper signs in the exponential expressions in Equations (4a-4c). With these coefficients, the tunnel passing coefficient can be recalculated. We have calculated these coefficients, but there is no need to give them here. Because, as can be seen below, these coefficients are not needed to calculate the scattering cross sections. Now, let us explain this situation below:

The terms with A_1 and B_1 in the functions given in Equation (4a) representing the outgoing and incoming waves respectively at the point $r = r_1$. So, the r_1 value is obtained by solving the equation $E = U_b(r)$. Thus, the wave functions arriving and scattered at the $r = r_1$ and passing through to the point $r = r_2$ are as follows:

$$C_s(r_1) = A_1\, e^{\pm Q_1(r_1)}, \ \text{(scattering amplitude)};$$
$$R_s(r) = \frac{F_s(r)}{r} = C_s(r_1)\frac{e^{i\,K\,r}}{r} = A_1\, e^{\pm Q_1(r_1)}\ \frac{e^{i\,K\,r}}{r} \tag{7a}$$

$$C_c(r_1) = B_1\, e^{\pm Q_1(r_1)}, \ \text{(incoming amplitude)};$$
$$R_c(r) = \frac{F_c(r)}{r} = C_c(r_1)\frac{e^{-i\,K\,r}}{r} = B_1\, e^{\pm Q_1(r_1)}\ \frac{e^{-i\,K\,r}}{r} \tag{7b}$$

$$C_p(r_2) = A_3\, e^{\pm Q_3(r_2)}, \ \text{(passing amplitude)};$$
$$R_p(r) = \frac{F_p(r)}{r} = C_p(r_2)\frac{e^{-\,i\,K\,r}}{r} = A_3\, e^{\pm Q_3(r_2)}\ \frac{e^{-\,i\,K\,r}}{r} \tag{7c}$$

The functions (7a-7c) represent scattering, incoming and transmission waves, respectively.

Calculation of Particle Currents

Using the (7a-7c) functions and the equation $J(r) = \frac{\hbar}{2\,m\,i}\left[R^*(r)\frac{dR(r)}{dr} - R(r)\frac{dR^*(r)}{dr}\right]$, the current-density expression, the current densities for these three functions are found as follows:

$$J_s = \frac{1}{r_1^2}\frac{\hbar\,K}{m}C_sC_s^* = \frac{1}{r_1^2}\frac{\hbar\,K}{m}\ |C_s|^2 \qquad \text{(scattering current density)} \tag{8a}$$

$$J_c = \frac{1}{r_1^2}\frac{\hbar\,K}{m}C_cC_c^* = \frac{1}{r_1^2}\frac{\hbar\,K}{m}\ |C_c|^2 \qquad \text{(incoming current density)} \tag{8b}$$

$$J_p = \frac{1}{r_1^2}\frac{\hbar\,K}{m}C_pC_p^* = \frac{1}{r_1^2}\frac{\hbar\,K}{m}\ \left|C_p\right|^2 \qquad \text{(passing current density)} \tag{8c}$$

CALCULATIONS OF SCATTERING CROSS-SECTIONS

Calculation of Differential Elastic Scattering Cross-Section

The probability per unit differential surface of a sphere of radius r_1, that an incident particle is scattered into the differential surface area on the sphere of radius r_1 , $dS = r_1^2\, d\Omega$, $[d\Omega = \sin(\theta)\, d\theta\, d\phi\,]$ is expressed as the ratio of the scattered current to the incident current, that is:

$$\frac{d\sigma_s}{dS} = \frac{d\sigma_s}{r_1^2 d\Omega} = \frac{J_s(r_1)}{J_c(r_1)} \quad \rightarrow \quad \frac{d\sigma_s}{d\Omega} = \frac{J_s(r_1)}{J_c(r_1)}\, r_1^2 \tag{9}$$

The differential elastic cross-section can be expressed in a simple form by putting Equations (8) into Equation (9) as follows:

$$\frac{d\sigma_s}{d\Omega} = \frac{C_s(r_1)\, C_s^*(r_1)}{C_c(r_1)\, C_c^*(r_1)}\, r_1^2 = \frac{|C_s(r_1)|^2}{|C_c(r_1)|^2}\, r_1^2 = \frac{J_s(r_1)}{J_c(r_1)}\, r_1^2 \tag{10}$$

Since the scattering is azimuthally symmetrical, the angle ϕ can be integrated out so that the expression given in Equation (10) can be written as follows:

$$\frac{d\sigma_s}{d\theta} = 2\,\pi\, \frac{J_s(r_1)}{J_c(r_1)}\, r_1^2 \sin(\theta) = 2\,\pi \frac{|C_s(r_1)|^2}{|C_c(r_1)|^2}\, r_1^2 \sin\,(\theta \tag{11}$$

The expression (11) shows the elastic scattering differential cross sections in the angle $d\theta$ which is usually measured experimentally.

Calculation of Differential Inelastic or Reaction (No-Elastic) Cross-Section

Differential reaction (capture of particle, emission of particle, inelastic collision...) cross- section per the solid angle can be found through the difference between the incoming current and the outgoing current divided by the former. By analogy with Equation (11), the differential reaction cross-section can be expressed as follows:

$$\frac{d\sigma_r}{d\theta} = 2\,\pi\, \frac{[J_c(r_1) - J_s(r_1)]}{J_c(r_1)}\, r_1^2 \sin(\theta) = 2\,\pi \frac{\left[|C_c(r_1)|^2 - |C_s(r_1)|^2\right]}{|C_c(r_1)|^2}\, r_1^2 \sin(\theta) \tag{12}$$

Calculation of Total Cross Sections

The total elastic scattering cross section is the total probability to be elastic scattered in any direction and it can be determined through the integral of differential cross-section given in Equation (11) as follows:

$$\sigma_s = \int d\sigma_s = \int \frac{d\sigma_s}{d\Omega} d\Omega = \iint \frac{J_s(r_1)}{J_c(r_1)}\, r_1^2 \sin(\theta)\, d\,\theta\, d\phi = 4\,\pi\, r_1^2\, \frac{J_s(r_1)}{J_c(r_1)} = 4\,\pi\, r_1^2 \frac{|C_s(r_1)|^2}{|C_c(r_1)|^2} \tag{13}$$

By analogy with Equation (11), the total reaction cross-section can be expressed as follows:

$$\sigma_r = 4\,\pi\, r_1^2\, \frac{[J_c(r_1) - J_s(r_1)]}{J_c(r_1)} = 4\,\pi\, r_1^2\, \left(1 - \frac{J_s(r_1)}{J_c(r_1)}\right) = 4\,\pi\, r_1^2\, \left(1 - \frac{|C_s(r_1)|^2}{|C_c(r_1)|^2}\right) = 4\,\pi\, r_1^2 - \sigma_s \tag{14}$$

In Equation (12), it is seen that if $J_s(r_1) = J_c(r_1)$, then $\sigma_r = 0$, full-elastic scattering; if $J_s(r_1) > J_c(r_1)$, then $\sigma_r < 0$, it is taken out of the particle from the target (emission of particle from target) and if $J_s(r_1) < J_c(r_1)$, then $\sigma_r > 0$, it is captured (absorbed) the particle by the target.

The total scattering cross-section, including all process [elastic plus reaction (all no-elastic events)]:

$$\sigma_t = \sigma_s + \sigma_r = 4\,\pi\, r_1^2\, \frac{J_s(r_1)}{J_c(r_1)} + 4\,\pi\, r_1^2\, \frac{[J_c(r_1) - J_s(r_1)]}{J_c(r_1)} = 4\,\pi\, r_1^2 \tag{15}$$

Then, the cross-sections $\sigma_s, \sigma_r, \sigma_t$ can be expressed through the $C_s(r_1)$ coefficient given above.

RELATIONSHIP BETWEEN SCATTERING CROSS SECTIONS AND POTENTIAL BARRIER TRANSMİSSİON COEFFICIENT (TRANSMISSION COEFFICIENT) T

It is also possible to easily calculate the scattering cross-sections in terms of the potential barrier transmission coefficient (T), which are calculated by the amplitudes of the waves and given by the formulas (13-15). But there is no need to calculate amplitudes and particle currents here. Because the scattering cross sections can be easily calculated based on the coefficient of crossing the potential barrier. Now, let us calculate the scattering cross sections depending on the coefficient of crossing the potential barrier.

Incoming particle current J_c, elastic scattering particle current J_s, transmission or remaining in the potential barrier (inelastic or reaction) particle current J_r ; let the total elastic scattering cross section σ_s , the total reaction cross section σ_r, and the total scattering cross section σ_t be. By putting $J_r = T\, J_c$ in expressions (13-15), the following expressions are obtained:

$$\sigma_s = 4\,\pi\, r_1^2\, \frac{J_c - J_r}{J_c} = 4\,\pi\, r_1^2 \left(1 - \frac{J_r}{J_c}\right) = 4\,\pi\, r_1^2 \left(1 - \frac{T\, J_c}{J_c}\right) = 4\,\pi\, r_1^2\, (1 - T\,) \tag{16}$$

$$\sigma_r = 4\,\pi\, r_1^2\, \frac{J_c - J_s}{J_c} = 4\,\pi\, r_1^2\, \frac{J_c - (J_c - J_r)}{J_c} = 4\,\pi\, r_1^2\, \frac{J_r}{J_c} = 4\,\pi\, r_1^2\, \frac{T\, J_c}{J_c} = 4\,\pi\, r_1^2\, T \tag{17}$$

$$\sigma_t = \sigma_s + \sigma_r = 4\,\pi\, r_1^2 \tag{18}$$

(J_c, J_s , J_r) are incoming, scattering, reaction currents, respectively. Here, T is the tunneling probability coefficient (or transmission coefficient of potential barrier) is given by the following formula [1-6]:

$$T = \frac{2}{\cosh[2\, K\, d] + \cos(2\, P)}$$

Here, the width of the potential barrier $d = r_1 - r_2$, $K = m_1\sqrt{|E|}$, $m_1 = \sqrt{2\,m}/\hbar$, E energy, and $P = Q_2(r_1) - Q_2(r_2)$. If $Q_2(r)$ is pair function, $P = 0$. If $Q_2(r)$ is not pair function, $P = \mathrm{Real}[Q_2(r_1) - Q_2(r_2)]$ zero or approximately zero can take. So, if $P = 0$, the following coefficient of transmission is obtained:

$$T(d_2) = \frac{2}{1 + \cosh[2\, K(E) d_2(E)]},\ d_2 = r_1 - r_2\ .\ \text{If}\ \ d_2 = 0,\ \text{then}\ T = 1 \tag{19}$$

It is easier to calculate T, which obviously does not depend on the shape of the potential barrier. It can be seen from (16-18) that the scattering cross sections depend on the penetration coefficient $T(d_2)$ and r_1. Calculating this coefficient T is easier than calculating amplitudes. In fact, the T coefficient is also obtained by calculating the scattering amplitudes described above. As a result, the scattering cross sections are simply.

$$\sigma_s = 4\,\pi\, r_1^2\, 10\left[1 - T\big(d_2(E)\big)\right] = 40\,\pi\, r_1^2\, \left[1 - T\big(d_2(E)\big)\right] \tag{20}$$

$$\sigma_r = 4\,\pi\, r_1^2\, 10\, T\big(d_2(E)\big) = 40\,\pi\, r_1^2\, T\big(d_2(E)\big) \tag{21}$$

$$\sigma_t = \sigma_s + \sigma_r = 4\,\pi\,r_1^2 * 10 = 40\,\pi\,r_1^2 \tag{22}$$

If r_1 and d_2 are taken as fermi (fm) in equations (20-22), the scattering cross sections will be millibarn (mb). The factor of 10 in these equations comes from taking such units. It can be seen from these formulas given in Equations (20-22) that the scattering cross-sections (σ_s, σ_r and σ_t) depend on the total energy E [with $K(E)$, $r_1(E)$ and $d_2(E)$]. Here, r_1 can be considered as impact or collision parameter, classically.

EXAMPLES OF THE CALCULATION OF SCATTERING CROSS-SECTION

Model Potentials and Their Some Ingredients

To calculate a scattering cross-section, a model potential should be considered. We consider the potential for scattering in Figure 1. As can be seen from the (20-22) formulas, wave functions are not needed to calculate the scattering cross sections. Therefore, there is no need to know the exact shapes of the potentials. It is sufficient to calculate only the r_1 and r_2 coordinates. Here as example, we consider two cases: **(1)** Rectangle potential as U_3 potential segment, and barrier potential U_b as U_2 potential segment (centrifugal potential plus Coulomb potential). **(2)** Harmonic oscillator potential as U_3 potential segment, and barrier potential U_b as potential U_2 (centrifugal potential plus Coulomb potential). The potential zones are defined in Figure 1. The scattering affects only relative motion. The scattering cross-section of the incoming (incident or projectile) particle depends on the relative energy $E_r = M_t E_L/(M_p + M_t)$, where M_p and M_t respectively mass of incident (projectile) and target particles; E_L, Laboratory energy; and E_r, relative energy.

Case 1. Rectangle Potential As U_3 Potential Segment and Barrier Potential U_b As U_2 Potential Segment

We have taken $r_2 = R_0\left(A_p^{1/3} + A_t^{1/3}\right)$ in potential segment U_3, and $U_b(r) = b/r^2 + C_c/r$ as potential segment U_2, where A_p and A_t are the mass numbers of the projectile and the target, respectively. R_0 is a parameter. The positive root of the equation $U_2(r) = U_b(r) = E_r$ is $r_{11} = C_c + \sqrt{C_c^2 + 4\,b\,E_r}/(2\,E_r)$. Then, $r_1 = r_2 + r_{11}$. So, $r_1 - r_2 = d_2(E) = r_{11}$ is the width of the potential barrier. The zones and r_k, $(k = 1, 2)$ values are shown in Figure 1.

$V_S(r) = b/r^2$, centrifugal potential; $b = \hbar^2\,J\,(J+1)/(2\,M_r)$, M_r reduced mass, J relative total angular momentum.

$V_c(r) = C_c/r$, Coulomb potential, $C_c = \left(Z_p e\right)(Z_t e) = Z_p Z_t e^2$, Z_p and Z_t are charge number of projectile and target, respectively. And e is the unit electric charge.

It is seen from Equations (20-22) that σ_t depends only on the r_1 value, although the σ_s and σ_r cross sections depend on the r_1 and T quantities. The r_2 quantity depends on the R_0 parameter. Even though σ_s and σ_r have changed with the T and r_1, $\sigma_t = \sigma_s + \sigma_r$ depend on only r_1. In other words, the total cross-section does not depend on the T. So, R_0 parameter can be obtained from the solution of the equation $4\pi r_1^2 * 10 = \sigma_t^{exp}$ as follows, (σ_t^{exp} is experimental total cross-section):

$$R_0 = \frac{-\sqrt{\pi}\,10\left[C_c + \sqrt{C_c^2 + 4\,b\,E_r}\right] + E_r\sqrt{10\,\sigma_t^{exp}}}{20\,E_r\sqrt{\pi}\left[\sqrt[3]{A_p} + \sqrt[3]{A_t}\right]}$$

Table 1. $[He\ [2\ ,3]\ +\ Xn\ [Z\ ,N]$ Cross-sections comparison with those measured (case 1)

$Xn[Z,N]$ (Target)	$E_L(MeV)$ He (2,3)	$r_2(fm)$	$r_1(fm)$	$d_2\ (fm)$	$T(d_2)$	σ_s^{cal} (mb)	σ_r^{cal} (mb)	σ_t^{cal} (mb)	σ_t^{exp} (mb)
Be (4,9)	96.4	2.13126	2.53101	0.399748	0.349472	524	281	805	805
Be (4,9)	137.8	1.98812	2.30905	0.320928	0.489133	342	328	670	670
Be (4,9)	167.3	1.94451	2.23016	0.285641	0.449812	344	281	625	625
C (6,12)	96.4	2.11525	2.53885	0.423602	0.274485	588	222	810	810
C (6,12)	137.8	2.04314	2.37697	0.333830	0.42318	410	300	710	710
C (6,12)	167.3	1.9711	2.26556	0.294453	0.389204	394	251	645	645
O (8,16)	96.4	2.33304	2.78546	0.452419	0.108278	869	106	975	975
O (8,16)	137.8	2.25012	2.60079	0.350669	0.315005	582	268	850	850
O (8,16)	167.3	2.21638	2.52313	0.306753	0.330808	535	265	800	800
Si (14,28)	96.4	2.57607	3.15392	0.577846	0.058062	1177	73	1250	1250
Si (14,28)	137.8	2.5937	3.02513	0.431427	0.234463	880	270	1150	1150
Si (14,28)	167.3	2.54117	2.91119	0.370019	0.184194	869	196	1065	1065
Ca (20,40)	96.4	2.56153	3.28976	0.728229	0.006241	1352	8	1360	1360
Ca (20,40)	137.8	2.65994	3.19154	0.531601	0.095339	1158	122	1280	1280
Ca (20,40)	167.3	2.67211	3.12222	0.450103	0.091308	1113	112	1225	1225
Ni (28,58)	96.4	2.72392	3.66723	0.943309	0.000722	1689	1	1690	1690
Ni (28,58)	137.8	2.85735	3.53464	0.677288	0.032315	1519	51	1570	1570
Ni (28,58)	167.3	2.85255	3.42022	0.567666	0.031998	1423	47	1470	1470
Ni (28,60)	96.4	2.72005	3.6618	0.941755	0.000605	1684	1	1685	1685
Ni (28,60)	137.8	2.90321	3.57938	0.676173	0.032315	1558	52	1610	1610
Ni (28,60)	167.3	2.89971	3.46644	0.566731	0.031998	1462	48	1510	1510
Sn (50,112)	96.4	2.54198	4.1122	1.570220	$1.6668*10^{-6}$	2125	0	2125	2125
Sn (50,112)	137.8	2.95942	4.06843	1.109010	0.001269	2077	3	2080	2080
Sn (50,112)	167.3	3.14889	4.06843	0.919543	0.001360	2077	3	2080	2080
Sn (50,116)	96.4	2.65792	4.22672	1.5688	$1.20939*10^{-6}$	2245	0	2245	2245
Sn (50,116)	137.8	3.02832	4.13632	1.108	0.00126874	2147	3	2150	2150
Sn (50,116)	167.3	3.21761	4.13632	0.918711	0.00136031	2147	3	2150	2150
Sn (50,118)	96.4	2.72861	4.29674	1.56813	$1.0125*10^{-6}$	2320	0	2320	2320
Sn (50,118)	137.8	3.12389	4.23142	1.10753	0.00009892	2250	0	2250	2250
Sn (50,118)	167.3	3.24676	4.16508	0.918316	0.00136031	2177	3	2180	2180
Sn (50,120)	96.4	2.69673	4.26421	1.56747	$1.20054*10^{-6}$	2285	0	2285	2285
Sn (50,120)	137.8	3.10551	4.21257	1.10707	0.00126874	2227	3	2230	2230
Sn (50,120)	167.3	3.24715	4.16508	0.917934	0.00136031	2177	3	2180	2180
Sn (50,124)	96.4	2.74438	4.31061	1.56623	$1.20735*10^{-6}$	2335	0	2335	2335
Sn (50,124)	137.8	3.09693	4.20312	1.10619	0.00126874	2217	3	2220	2220
Sn (50,124)	167.3	3.22872	4.14593	0.917206	0.00136031	2157	3	2160	2160
Pb(82,208)	96.4	2.18335	4.69075	2.50741	$1.4303*10^{-10}$	2765	0	2765	2765
Pb (82,208)	137.8	3.00165	4.76231	1.76066	$9.54594*10^{-6}$	2850	0	2850	2850
Pb (82,208)	167.3	3.28314	4.73718	1.45403	0.00001124	2820	0	2820	2820

Thus, r_1 and r_2 values are calculated, and d_2 and $T(d_2)$ values are found. With these values, σ_s , σ_r and σ_t values are calculated. The mass values of the nuclei and some conversion factors and the values of the universal constants used were taken from [7, 8]. We have taken in the calculations the following values: $e^2 = 1.439976$ MeV fm ; $M_u = 931.502$ MeV/c^2 ; $\hbar$ c = 197.329 MeV fm . The cross-sections at three different energies of He[2, 3] have been calculated for many different targets. The calculated cross-sections have been compared with those measured taken from [9-11]. The comparisons are made in the way that is described above and given in Table 1. In this table: Xn [Z, N] target; E_L laboratory energy of projectile; σ_r^{cal}, σ_s^{cal} , σ_t^{cal} calculated reaction, elastic total cross-sections; σ_t^{exp} experimental measured total cross-section. There is no need for relative angular momentums here. Because the individual total angular momentum of the target particles, except Be (4,9), is zero.

Case 2. Harmonic Oscillator Potential As U_3 Potential, Barrier Potential U_b As Potential U_2 (Centrifugal Potential Plus Coulomb Potential).

We have taken the harmonic oscillator potential $V(r) = \frac{1}{2} M_r \omega^2 r^2$ as potential segment U_3, and $U_b(r) = b/r^2 + C_c/r$ as potential segment U_2, where M_r and ω are the reduced mass of the projectile and the target, and a parameter, respectively. The positive root of the equation $V(r) = E_r$ is $r_2 = \sqrt{2 E_r / (M_r \omega^2)}$, and the positive root of the equation $U_b(r) = E_r$ is $r_{11} = C_c + \sqrt{C_c^2 + 4 b E_r}/(2 E_r)$. Then, $r_1 = r_2 + r_{11}$. So, $r_1 - r_2 = d_2(E) = r_{11}$ is the width of the potential barrier. The zones and r_k , $(k = 1, 2)$ values are shown in Figure 1.

$V_S(r) = b/r^2$, centrifugal potential ; $b = \hbar^2 J (J + 1)/(2 M_r)$, M_r reduced mass, J relative total angular momentum.

$V_c(r) = C_c/r$, Coulomb potential, $C_c = (Z_p e)(Z_t e) = Z_p Z_t e^2$, Z_p and Z_t are charge number of projectile and target, respectively. And e is the unit electric charge.

It is seen from Equations (20-22) that σ_t depends only on the r_1 value, although the σ_s and σ_r cross sections depend on the r_1 and T quantities. The r_2 quantity depends on the ω parameter. Even though σ_s and σ_r have changed with the T and r_1 , $\sigma_t = \sigma_s + \sigma_r$ depend on only r_1 . In other words, the total cross-section does not depend on the T . So, ω parameter can be obtained from the solution of the equation $4\pi r_1^2 * 10 = \sigma_t^{exp}$ as follows (σ_t^{exp} is experimental total cross-section):

$$\omega = \frac{4\left(-5\sqrt{2}\ E_r\left(C_c + \sqrt{C_c^2 + 4\, b\, E_r}\right)\sqrt{E_r M_r}\ \pi + \sqrt{5}\,\pi\sqrt{E_r^5\, M_r\, \sigma_t^{exp}}\right)}{M_r\left(20\,(C_c^2 + 2\, b\, E_r)\,\pi + 20\, C_c\sqrt{C_c^2 + 4\, b\, E_r}\,\pi - E_r^2\,\sigma_t^{exp}\right)}$$

After that, the procedures in case 1 were performed exactly and the results are given in Table 2. Experimental cross section values were taken from [9, 11].

Table 2. [He $[2\,,3]$ + Xn $[Z\,,N]$ Cross-sections comparison with those measured (case 2)

$\mathbf{Xn\ (Z,N)}$ $\mathbf{(Target)}$	$\mathbf{E_L(MeV)}$ **He (2,3)**	$\mathbf{r_2(fm)}$	$\mathbf{r_1(fm)}$	$\mathbf{d_2\ (fm)}$	$\mathbf{T(d_2)}$	$\boldsymbol{\sigma_s^{cal}}$ $\mathbf{(mb)}$	$\boldsymbol{\sigma_r^{cal}}$ $\mathbf{(mb)}$	$\boldsymbol{\sigma_T^{cal}}$ $\mathbf{(mb)}$	$\boldsymbol{\sigma_T^{exp}}$ $\mathbf{(mb)}$
Be (4,9)	96.4	213.126	253.101	0.399748	0.349472	524	281	805	805
Be (4,9)	137.8	198.812	230.905	0.320928	0.489133	342	328	670	670
Be (4,9)	167.3	194.451	223.016	0.285641	0.449812	344	281	625	625
C (6,12)	96.4	211.525	253.885	0.423602	0.162172	679	131	810	810
C (6,12)	137.8	204.314	237.697	0.33383	0.42318	410	300	710	710
C (6,12)	167.3	19.711	226.556	0.294453	0.389204	394	251	645	645
O (8,16)	96.4	233.304	278.546	0.452419	0.108278	869	106	975	975
O (8,16)	137.8	225.012	260.079	0.350669	0.358975	545	305	850	850
O (8,16)	167.3	221.638	252.313	0.306753	0.330808	535	265	800	800
Si (14,28)	96.4	257.607	315.392	0.577846	0.0331079	1209	41	1250	1250
Si (14,28)	137.8	25.937	302.513	0.431427	0.116715	1016	134	1150	1150
Si (14,28)	167.3	254.117	291.119	0.370019	0.184194	869	196	1065	1065
Ca (20,40)	96.4	256.153	328.976	0.728229	0.00624113	1352	8	1360	1360
Ca (20,40)	137.8	265.994	319.154	0.531601	0.0953393	1158	122	1280	1280
Ca (20,40)	167.3	267.211	312.222	0.450103	0.0913077	1113	112	1225	1225
Ni (28,58)	96.4	272.392	366.723	0.943309	0.000722156	1689	1	1690	1690
Ni (28,58)	137.8	285.735	353.464	0.677288	0.0323153	1519	51	1570	1570
Ni (28,58)	167.3	285.255	342.022	0.567666	0.0319982	1423	47	1470	1470
Ni (28,60)	96.4	272.005	36.618	0.941755	0.000604864	1684	1	1685	1685
Ni (28,60)	137.8	290.321	357.938	0.676173	0.0323153	1558	52	1610	1610
Ni (28,60)	167.3	289.971	346.644	0.566731	0.0319982	1462	48	1510	1510
Sn (50,112)	96.4	254.198	41.122	157.022	$1.6668*10^{-6}$	2125	0	2125	2125
Sn (50,112)	137.8	295.942	406.843	110.901	0.00126874	2077	3	2080	2080
Sn (50,112)	167.3	314.889	406.843	0.919543	0.00136031	2077	3	2080	2080
Sn (50,116)	96.4	265.792	422.672	15.688	$1.2094*10^{-6}$	2245	0	2245	2245
Sn (50,116)	137.8	302.832	413.632	1.108	0.00126874	2147	3	2150	2150
Sn (50,116)	167.3	321.761	413.632	0.918711	0.00136031	2147	3	2150	2150
Sn (50,118)	96.4	272.861	429.674	156.813	$1.2008*10^{-6}$	2320	0	2320	2320
Sn (50,118)	137.8	312.389	423.142	110.753	0.00126874	2247	3	2250	2250
Sn (50,118)	167.3	324.676	416.508	0.918316	0.00136031	2177	3	2180	2180
Sn (50,120)	96.4	269.673	426.421	156.747	$1.2005*10^{-6}$	2285	0	2285	2285
Sn (50,120)	137.8	310.551	421.257	110.707	0.00126874	2227	3	2230	2230

Sn (50,120)	167.3	324.715	416.508	0.917934	0.00136031	2177	3	2180	2180
Sn (50,124)	96.4	274.438	431.061	156.623	$1.2074*10^{-6}$	2335	0	2335	2335
Sn (50,124)	137.8	309.693	420.312	110.619	0.00126874	2217	3	2220	2220
Sn (50,124)	167.3	322.872	414.593	0.917206	0.00136031	2157	3	2160	2160
Pb (82,208)	96.4	218.335	469.075	250.741	$1.4303*10^{-10}$	2765	0	2765	2765
Pb (82,208)	137.8	300.165	476.231	176.066	$9.5459*10^{-6}$	2850	0	2850	2850
Pb (82,208)	167.3	328.314	473.718	145.403	0.00001124	2820	0	2820	2820

When Tables 1 and 2 Are Examined, The Following Can Be Said:
It is not necessary to know the exact form of the U_3 potential segment in the calculation of the scattering cross sections. It doesn't matter where the coordinate start is taken. There is no need to calculate wave functions. It is sufficient to find the width of the barrier potential causing the scattering and the distance from the potential center of the point where the incoming current touches the potential barrier. Thus, it becomes very easy to calculate the scattering cross sections.

$\mathrm{Sn}(50, 112-124)$ nuclear nuclei are a very good barrier for the energy 96.4 MeV of incoming current, and $\mathrm{Pb}(82, 208)$ nuclear nuclei for all three-energy of incoming current.

WHEN THE SCATTERING CROSS-SECTIONS ARE KNOWN, THE CALCULATION OF THE BARRIER WIDTH AND BARRIER TRANSMİSSİON COEFFICIENT

By measuring the scattering cross sections, the width of the potential barrier and the transmission coefficient of barrier can be calculated. From equations (20-22), we get the following values:

$$r_1 = \frac{\sqrt{\sigma_r + \sigma_s}}{2\sqrt{10\,\pi}} \quad ;$$

$$T = \frac{\sigma_r}{\sigma_r + \sigma_s} \quad ;$$

$$r_2 = r_1 - \frac{\operatorname{arccosh}[(2-T)/T]}{2\,K * 10^{13}} \quad ;$$

$$d_2(E_r) = \frac{\operatorname{arccosh}[(2-T)/T]}{2\,K * 10^{13}}$$

In these equations, the desired values are obtained by taking the experimentally measured cross-sections:

$T\big(d_2(E_r)\big) = \frac{\sigma_r^{exp}}{\sigma_r^{exp}+\sigma_s^{exp}}$, obstacle passing coefficient ; $d_2(E_r) = \frac{\operatorname{arccosh}[(2-T)/T]}{2\,K*10^{13}}$, Obstacle width. The barrier potential widths and barrier transmission coefficients calculated for many thermal neutron cross sections are given in Table 3. Experimental cross section values were taken from [9].

Table 3. $[\mathbf{n}\,(\mathbf{0}\,,\mathbf{1})\;+\;\mathbf{Xn}\,(\mathbf{Z}\,,\mathbf{N})]$**, when the scattering cross-sections are known, the calculation of the barrier width and barrier transmission coefficient for thermal neutrons.**

Neucleus Xn (Z, N)	σ_S^{exp} (mb)	σ_R^{exp} (mb)	σ_T^{exp} (mb)	r_2 (fm)	r_1 (fm)	d_2 (fm)	$T\,(d_2)$
n (0,1)	20491	332.6	20823.6	12.8728	12.8728	1.58781x10^{-8}	0.015972
H (1,1)	20491	332.6	20823.6	12.8728	12.8728	1.58781x10^{-8}	0.015972
H (1,2)	3390	0.519	3390.52	5.19431	5.19431	2.93019x10^{-8}	0.000153
Be (4,9)	6151	7.6	6158.6	7.00061	7.00061	1.74644x10^{-8}	0.001234
C (6,12)	4746	3.53	4749.53	6.14781	6.14781	1.37499x10^{-8}	0.000743
O (8,16)	3761	0.19	3761.19	5.47089	5.47089	1.76016*10^{-8}	0.000051
Si (14,28)	1992	177	2169	4.15456	4.15456	5.89164x10^{-9}	0.081604
Ca (20,40)	3010	410	3420	5.21685	5.21685	5.13741x10^{-9}	0.119883
Ni (28,58)	25300	4600	29900	15.4252	15.4252	4.6877x10^{-9}	0.153846
Ni (28,60)	980	2900	3880	5.55662	5.55662	1.61904x10^{-9}	0.747423
Sn (50,112)	4909	626	5535	6.63673	6.63673	5.13281x10^{-9}	0.113098
Sn (50,114)	4600	115	4715	6.12542	6.12542	7.38953x10^{-9}	0.024390
Sn (50,118)	4260	220	4480	5.97082	5.97082	6.3536x10^{-9}	0.049107
Sn (50,120)	5170	140	5310	6.50043	6.50043	7.27166x10^{-9}	0.026365
Sn (50,124)	4410	134	4544	6.01332	6.01332	7.10577x10^{-9}	0.029489
Te (52,124)	3800	6800	10600	9.18434	9.18434	2.00611x10^{-9}	0.641509
Ba (56,130)	3420	1200	4620	6.0634	6.0634	3.75812x10^{-9}	0.259740
Ce 58,140)	2830	570	3400	5.20157	5.20157	4.47176x10^{-9}	0.167647
Nd 60,142)	7700	18700	26400	14.4943	14.4943	1.75215x10^{-9}	0.708333
Nd 60,148)	4000	2500	6500	7.19203	7.19203	3.06436x10^{-9}	0.384615

When Table 3 is examined, it can be seen that $d_2 \neq 0$ and $T \neq 1$ are for thermal neutrons, although $r_1 = r_2$. This situation can be explained as follows: (1) The calculation of r_1 ve r_2 may be due to their precession. (2) There may be a barrier potential other than the centrifugal potential in the structure of the target nuclei.

CONCLUSION

The calculation of cross sections through solution of radial SE (RSE) by the partial wave expansion is exceedingly difficult. In many cases, some approximations are needed for these kinds of solutions. For these reasons, there is no simple scattering cross-section formula. However, here, simple scattering cross-section formulas were obtained without any approximation. These formulas will provide great convenience to those who do research on these issues. It will provide great convenience especially in material physics research. Examples here are taken from nuclear physics, but this method can also be used in other branches of physics. In the present study, firstly, differential elastic scattering, inelastic (or reaction) scattering and total cross-sections have been calculated without using any approximation. These calculations have been performed using a simple method, improved for the solution of RSE, for an incident particle being in a central field of any form. We have obtained the general formulas of the scattering amplitudes and elastic, inelastic (no-elastic) and total scattering cross-sections. Secondly, we have made some applications. In these applications, the potentials have been assumed to have two shapes plus centrifugal and Coulomb potentials. Calculations with these are not easy by the partial wave expansion. However, it is quite easy to make calculations with our method. With these potentials, the elastic, inelastic (neutron radiative capture, etc.), and total scattering cross-sections of different targets have been calculated. The calculated results have been compared with experimental results. The results calculated have given satisfactory agreement with the available experimental results. Then, when the experimental elastic and reaction cross sections are known, we calculated the width of the potential barrier and the coefficient of transmission the potential barrier. By taking thermal neutrons as incoming particles, the potential barrier transmission coefficients and barrier widths were calculated for many nuclei.

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